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GÖTTINGEN

The Atiyah class and ideal systems

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Friday Fish Seminar — Utrecht, 14.8.2020

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The Bott connection

The Bott connection

(A, J) is a "Lie pair".

Let $A \to M$ be a Lie algebroid and $J \subseteq A$ a subalgebroid. The **Bott connection** is the flat J-connection on A/J defined by

$$\nabla^J \colon \Gamma(J) \times \Gamma(A/J) \to \Gamma(A/J), \qquad \nabla^J_j \bar{a} = \overline{[j, a]}.$$

$j \in \Gamma(J)$ $\qquad a \in \Gamma(A)$

- Flatness = Jacobi id.

$$R_{\nabla^J}(j_1, j_2)\bar{a} = \overline{\mathrm{Jac}_{[\cdot,\cdot]}(j_1, j_2, a)} = 0$$

Examples:

- A a Lie algebra $\mathfrak{g}$; J a subalgebra $\mathfrak{h}$

$$\nabla^{\mathfrak{h}} = \overline{\mathrm{ad}} : \mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$$

NB: if $\mathfrak{h}$ is an ideal $\mathfrak{i}$ then $\nabla^{\mathfrak{i}} = \overline{\mathrm{ad}} = 0$.

- More generally if $I \subseteq A$ is naive ideal

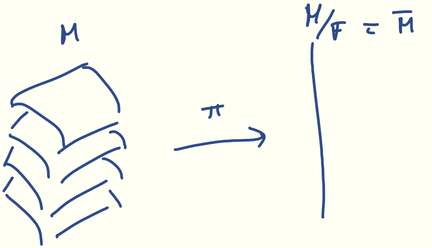
i.e. $[\Gamma(I), \Gamma(A)] \subseteq \Gamma(I)$ then $\nabla^I = 0$

$(\rightsquigarrow \rho(I) = 0)$

The Bott connection and foliations

$A = TM \to M$, $J = F \subseteq TM$ "foliation".

$\nabla^F : \Gamma(F) \times \Gamma(TM/F) \longrightarrow \Gamma(TM/F)$ $\qquad \nabla^F_X \bar{Y} = \overline{[X,Y]}$



$Y \in \mathfrak{X}(M)$ ∇^F-flat

$:\iff \bar{Y} \in \Gamma(TM/F)$ ∇^F-flat

$\iff [Y, \Gamma(F)] \subseteq \Gamma(F)$ (*)

$\iff \exists\, \tilde{Y} \in \mathfrak{X}(\bar{M})$ s.t.

$Y \sim_\pi \tilde{Y}$

F is "simple" if M/F is a manifold.

The Bott connection – properties

If $Y_1, Y_2 \in \mathfrak{X}(M)$ $\quad \nabla^F$- flat

$\Rightarrow [Y_1, Y_2]$ $\quad \nabla^F$- flat.

Why? Jacobi: $\quad \forall X \in \Gamma(F)$

$$\nabla^F_X \overline{[Y_1, Y_2]} = \overline{[X, [Y_1, Y_2]]}$$

$$\overset{\text{Jacobi}}{=} \underbrace{\overline{[\underbrace{[X, Y_1]}_{\in \Gamma(F)}, Y_2]}}_{\in \Gamma(F)} + \overline{[Y_1, [X, Y_2]]} = 0$$

Atiyah class of a Lie pair

Let (A, J) be a **Lie pair**. The **Atiyah class of the Lie pair** is a cohomology class

$$\alpha_J \in H^1(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J))).$$

If it vanishes, there exists an extension $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ of ∇^J such that

$$\bar{a}, \bar{b} \in \Gamma(A/J) \ \nabla^J\text{-flat} \qquad \Rightarrow \qquad \overline{\nabla_a b} \ \nabla^J\text{-flat}.$$

" ∇ transversal to ∇^J ".

If $A = TM$, $J = F$ simple

X, Y ∇^F-flat $\Rightarrow$ $\nabla_X Y$ projectable

X, Y projectable

$\rightarrow$ projects to a conn.

$\bar{\nabla} : \mathfrak{X}(\bar{M}) \times \mathfrak{X}(\bar{M}) \to \mathfrak{X}(\bar{M})$

Atiyah class of a Lie pair

Construction :

① Take $\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A)$ that extends ∇^J i.e.

$\nabla_a j \in \Gamma(J) \quad \forall a \in \Gamma(A), j \in \Gamma(J)$

and ($\overline{\nabla} : \Gamma(A) \times \Gamma(A/J) \to \Gamma(A/J)$ does $\overline{\nabla}_j \overline{a} = \nabla^J_j \overline{a}$ $\forall j \in \Gamma(J)$.)

$\left(\begin{array}{l} A = J \oplus K \\ \nabla^1 \ A\text{-conn. on } A/J \\ \overline{\nabla}_{j+k} \overline{a} = \nabla^1_k \overline{a} + \nabla^J_j \overline{a} \end{array} \right)$

② $\overline{a}, \overline{b}$ ∇^J-flat Want : $\forall j \in \Gamma(J)$

$$0 = \nabla^J_j \overline{\nabla_a b} = \nabla^J_j \overline{\nabla}_a \overline{b}$$

$$= \overline{\nabla}_j \overline{\nabla}_a \overline{b} - \overline{\nabla}_a \underbrace{\overline{\nabla}_j \overline{b}}_{=0} - \underbrace{\overline{\nabla}_{\underbrace{[j,a]}_{\in \Gamma(J)}} \overline{b}}_{=0} = R_{\overline{\nabla}}(j,a)\overline{b}$$

③ def. $\omega_\nabla \in \Omega^1(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J)))$

$\omega_\nabla(j, \overline{a}, \overline{b}) = R_{\overline{\nabla}}(j,a)\overline{b}$

$d_{\nabla^J} \omega_\nabla = 0$

$\to \alpha_J = [\omega_\nabla] \in H^1(J, \ldots$

④ if $\alpha_J = 0$ $\quad \omega_\nabla = d\phi \quad \phi \in \Gamma(\mathrm{Hom}(A/J, \mathrm{End}(A/J))) \to \nabla - \phi$ does it

Atiyah class of a Lie pair

- Chen, Stiénon, Xu 2016: *From Atiyah Classes to Homotopy Leibniz Algebras*.
- Laurent-Gengoux, Stiénon, Xu 2014: *Poincaré–Birkhoff–Witt isomorphisms and Kapranov dg-manifolds*.

The Atiyah class of a holomorphic vector bundle

Holomorphic vector bundles

Theorem (Kobayashi)

Let $E \to M$ be a complex vector bundle over a complex manifold M. Then E is a holomorphic vector bundle if and only if there exists a $\mathbb{C}$-linear connection

$J : TM \to TM$

$J_{\mathbb{C}} : TM_{\mathbb{C}} \to TM_{\mathbb{C}}$

$$D = D^{1,0} + D^{0,1} \colon \Gamma(TM_{\mathbb{C}}) \times \Gamma(E) \to \Gamma(E)$$

$TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$

eigenspaces of $J_{\mathbb{C}}$ to i and $-i$.

such that $D^{0,1}$ is flat.

$\to$ $D^{0,1} = \bar{\partial}$-operator

$\bar{\partial}_{\bullet} e = 0 \iff e \in \Gamma(E)$ holomorphic

$(TM_{\mathbb{C}}, T^{0,1}M)$ Lie pair.

Holomorphic connections

Let $E \to M$ be a holomorphic vector bundle. A $\mathbb{C}$-linear connection $\nabla \colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ is holomorphic if $\nabla_X e$ is holomorphic for $X \in \mathfrak{X}_{\mathcal{U}}(M)$ a local holomorphic vector field and $e \in \Gamma_U(E)$ a local holomorphic section.

$\rightarrow D : \Gamma(TM_{\mathbb{C}}) \times \Gamma(E) \to \Gamma(E)$

fn $\left.\begin{array}{ll} X \in \Gamma(T^{1,0}M) & \bar\partial\text{-flat.} \\ e \in \Gamma(E) & \bar\partial\text{-flat} \end{array}\right\} \Longrightarrow D_X e \ \ \bar\partial\text{-flat}$

$\rightsquigarrow \alpha_E \in H^1(T^{0,1}M, \mathrm{Hom}(T^{1,0}M, \mathrm{End}(E)))$

Atiyah class of the hol. VB E.

$\alpha_E = 0$ iff $\exists$ hol. conn.

Holomorphic Lie algebroids

A holomorphic Lie algebroid is a holomorphic vector bundle $A \to M$ with a Lie algebroid structure $(\rho, [\cdot\,,\cdot])$ such that

$$* \; [\mathcal{A}, \mathcal{A}] \subseteq \mathcal{A}, \quad \text{and} \quad \rho(\mathcal{A}) \subseteq \mathcal{X}.$$

$TM_{\mathbb{C}} = T^{0,1}M \oplus T^{1,0}M \qquad A_{\mathbb{C}} = A^{0,1} \oplus A^{1,0}$

$\bar\partial : \Gamma(T^{0,1}M) \times \Gamma(A^{1,0}) \to \Gamma(A^{1,0}) \qquad A^{1,0} \cong A_{\mathbb{C}} / A^{0,1}$

- $a, b \in \Gamma(A^{1,0})$ $\bar\partial$-flat $\Rightarrow$ $[a,b]$ $\bar\partial$-flat
- $a \in \Gamma(A^{1,0})$ $\bar\partial$-flat $\Rightarrow$ $[a,b] \in \Gamma(A^{0,1})$ $\forall\, b \in \Gamma(A^{0,1})$ $\quad *$
- $\rho_{\mathbb{C}}(A^{0,1}) \subseteq T^{0,1}M$
- $a \in \Gamma(A^{1,0})$ $\bar\partial$-flat $\Rightarrow$ $\rho_{\mathbb{C}}(a) \in \Gamma(T^{1,0}M)$ $\bar\partial$-flat

$\rightsquigarrow (T^{0,1}M, A^{0,1}, \bar\partial)$ infinitesimal ideal system in $A_{\mathbb{C}}$

Infinitesimal ideals and the Atiyah class

Infinitesimal ideals

(A, J) "infinitesimal ideal pair".

Definition (JL-Ortiz 14, Hawkins 07)

Let $(q\colon A \to M, \rho, [\cdot\,,\cdot])$ be a Lie algebroid, $F_M \subseteq TM$ an involutive subbundle, $J \subseteq A$ a subalgebroid over M such that $\rho(J) \subseteq F_M$ and ∇ a flat F_M-connection on A/J with the following properties:

1. If $a \in \Gamma(A)$ is ∇-flat, then $[a, j] \in \Gamma(J)$ for all $j \in \Gamma(J)$.
2. If $a, b \in \Gamma(A)$ are ∇-flat, then $[a, b]$ is also ∇-flat.
3. If $a \in \Gamma(A)$ is ∇-flat, then $\rho(a)$ is ∇^{F_M}-flat.

if $g(J) = F_M$ then 2 and 3 follow from 1.

The triple (F_M, J, ∇) **is an infinitesimal ideal system in** A**.**

1 is eq. to:

$$\nabla^J_j \bar{a} = \nabla_{g(j)} \bar{a} \qquad \forall j \in \Gamma(J) \quad \forall a \in \Gamma(A)$$

Infinitesimal ideals

$$
\begin{tikzcd}
\bar{A} = A/J\big/\nabla \arrow[r] & M/F_M =: \bar{M} \\
\underset{\text{kernel of } \pi}{J} \subseteq A \arrow[u, "\pi"] \arrow[r] & M \arrow[u, "\pi_M"']
\end{tikzcd}
$$

fibration of Lie algebroids

∇ tells us which sections of A project to sections of $\bar{A}$.

$\boxed{\text{Question: When is } (A, J) \text{ an ideal pair?}}$

in that case (F_M, J, ∇) is an "ideal system" in the sense of Mackenzie-Higgins.

$\uparrow$

then (A, J) is an "ideal pair".

Atiyah class of an infinitesimal ideal

Let (F_M, J, ∇^i) be an infinitesimal ideal in A. The **Atiyah class**

$$\alpha \in H^1(F_M, \mathrm{Hom}(TM/F_M, \mathrm{End}(A/J)))$$

of the infinitesimal ideal is a cohomology class that vanishes if and only if there exists an extension $\nabla\colon \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A)$ of ∇^i such that

$\nabla^i : \Gamma(F_M) \times \Gamma(A/J) \to \Gamma(A/J)$

$$\begin{cases} \bar{a} \in \Gamma(A/J) \ \nabla^i\text{-flat and} \\ \bar{X} \in \Gamma(TM/F_M) \ \nabla^{F_M}\text{-flat} \end{cases} \quad \Rightarrow \quad \overline{\nabla_X a} \ \nabla^i\text{-flat.}$$

Reducible algebroid

Theorem

Let (F_M, J, ∇) be an infinitesimal ideal in a Lie algebroid $A \to M$. If the quotient vector bundle $A' := (A/J)/\nabla \to M/F_M =: M'$ exists, then the Atiyah class of the infinitesimal ideal vanishes.

In other words...

Theorem

Let (F_M, J, ∇) be an infinitesimal ideal in a Lie algebroid $A \to M$. If (F_M, J, ∇) integrates to an ideal, then the Atiyah class of (F_M, J, ∇) vanishes.

NB: $F \subseteq TM$ inv. subb. $\Rightarrow$ (F, F, ∇^F) i.i.s. in A

There are examples of foliations F with $\alpha_F = 0$ but F is not simple. So the converse of these theorem is not true in general!

Foliated principal bundles

Foliated principal bundles

A principal G-bundle $\pi\colon P \to M$ is described infinitesimally by its Atiyah sequence

$$0 \to \mathfrak{g}_P \to \frac{TP}{G} \to TM \to 0.$$

$F_\Pi \subseteq TM$

A principal foliation on $\pi\colon P \to M$ is an involutive subbundle $F \subseteq TP$

- that is G-invariant; $T_p\Phi_g F(p) = F(pg)$ for all $p \in P, g \in G$,
- and with $F^\pi := F \cap T^\pi P$ of constant rank.

Molino, Karber – Tondeur 70's.

The infinitesimal ideals

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. Then there is an ideal $\mathfrak{i} \subseteq \mathfrak{g}$ such that

$$F^{\pi}(p) = \{x_P(p) \mid x \in \mathfrak{i}\} \quad \text{for all} \quad p \in P.$$

The associated bundle $\mathfrak{i}_P$ is a naive ideal in TP/G.

$(F_M, \mathfrak{i}_P, \nabla)$

$(F_M, F/G, \nabla)$

inf. ideal systems.

∇ defined by the Bott connection $\nabla^{F/G}$ on $(TP/F)/G$

The Atiyah class(es)

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & i_P \arrow[r] \arrow[d] & F/G \arrow[r] \arrow[d] & F_M \arrow[r] \arrow[d] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & \mathfrak{g}_P \arrow[r] \arrow[d] \arrow[dddr, dashed] & \frac{TP}{G} \arrow[r] \arrow[d] \arrow[dddr, dashed] & TM \arrow[r] \arrow[d] \arrow[dddr, dashed] \arrow[l, bend right, "s"'] & 0 \\
0 \arrow[r] & \left(\mathfrak{g}/\mathfrak{h}\right)_P \arrow[r] \arrow[d] & \frac{TP/F}{G} \arrow[r] \arrow[d] & TM/F_M \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & \\
 & 0 \arrow[r] & \mathfrak{g}_{\bar P} \arrow[r] & T\bar P \arrow[r] & T\bar M \arrow[r] & 0
\end{tikzcd}
$$

$\omega_s \in \Omega^2(M, \mathfrak{g}_P)$

$\rightsquigarrow \omega_s \in \Omega^2(F_M, i_P)$

$\rightsquigarrow \overline{\omega_s} \in \Omega^1(F_M, \mathrm{Hom}(TM/F_M, (\mathfrak{g}/\mathfrak{h})_P))$

$d_\nabla \overline{\omega_s} = 0$

$\alpha = [\omega_s] \in H^1(F_M, \mathrm{Hom}(TM/F_M, (\mathfrak{g}/\mathfrak{h})_P))$

Lie pairs and the Atiyah class

(A, J) a Lie pair; $F_M \subseteq TM$ involutive subbundle with $\rho(J) \subseteq F_M$ and a flat F_M-connection ∇ on A/J. Then

$$\rho^\star \colon \Omega^\bullet(F_M, \mathrm{Hom}(TM/F_M, \mathrm{End}(A/J))) \to \Omega^\bullet(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J)))$$

is defined by

$$(\rho^\star\omega)(j_1, \ldots, j_p)(\overline{a_1}, \overline{a_2}) = \omega(\rho(j_1), \ldots, \rho(j_p))(\overline{\rho(a_1)})(\overline{a_2}).$$

If $\nabla_{\rho(j)}\bar{a} = \nabla^J_j \bar{a}$ for all $j \in \Gamma(J)$, $a \in \Gamma(A)$, then

$$\mathbf{d}_{\nabla^J} \circ \rho^\star = \rho^\star \circ \mathbf{d}_{\nabla^{\mathrm{Hom}}}.$$

Theorem (JL 19)

If (F_M, J, ∇) is an infinitesimal ideal in A, then the image under $\rho^\star$ of its Atiyah class

$$\alpha \in H^1_{\mathbf{d}_{\nabla^{\mathrm{Hom}}}}(F_M, \mathrm{Hom}(TM/F_M, \mathrm{End}(A/J)))$$

is the Atiyah class

$$\alpha_J \in H^1_{\mathbf{d}_{\nabla^J}}(J, \mathrm{Hom}(A/J, \mathrm{End}(A/J)))$$

of the Lie pair.

Obstruction result.

Theorem (JL 19)

Let (A, J) be a Lie pair. If (A, J) is an ideal pair, then $\alpha_J = 0$.

NB: $(F_M, J, \nabla) \Leftrightarrow$ $F_A \subseteq TA$ involutive subbundle s.t.

$$\begin{tikzcd} F_A \arrow[r] \arrow[d] \arrow[dr, phantom, "J"] & A \arrow[d] \\ F_M \arrow[r] & M \end{tikzcd} \subseteq \begin{tikzcd} TA \arrow[r] \arrow[d] \arrow[dr, phantom, "A"] & A \arrow[d] \\ TM \arrow[r] & M \end{tikzcd}$$

double subalgebroid.

References

- *Foliated groupoids and infinitesimal ideal systems*, with Cristian Ortiz, "Indagationes Mathematicae " (2014), Volume 25, Number 5, 1019-1053.
- *VB-algebroids morphisms and representations up to homotopy*, with Thiago Drummond and Cristian Ortiz, "Differential Geometry and its Applications" (2015), Volume 40, 332-357.
- *Obstructions to representations up to homotopy and ideals*, arXiv:1905.10237, 2019.
- *Infinitesimal ideal systems and the Atiyah class*, arXiv:1910.04492, 2019, new version very soon!
- *Linear generalised complex structures*, with Malte Heuer, in preparation (2020).

* See also arXiv:1202.1378, Zambon & Zhu
for the NQ-geom. version of inf. ideals.

Thank you for your attention!