Universal centralizers and Poisson transversals

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G semisimple algebraic group of adjoint type over $\mathbb C$ $\operatorname{rank}(G)=I$ $\mathfrak g=\operatorname{Lie}\, G$

The regular locus of $\mathfrak g$ is

$$\mathfrak{g}^{\mathsf{r}} = \{ x \in \mathfrak{g} \mid \dim G^{\mathsf{x}} = I \}.$$

- this is the regular locus of the KKS Poisson structure
- x regular semisimple $\rightsquigarrow G^x$ is a maximal torus
- x regular nilpotent $\leadsto G^x$ is a abelian group $\cong \mathbb{C}^I$

Let $\{e, h, f\} \subset \mathfrak{g}$ be a regular \mathfrak{sl}_2 -triple.

Theorem (Kostant)

The principal slice

$$S = f + \mathfrak{g}^e \subset \mathfrak{g}^r$$

meets each regular G-orbit on \mathfrak{g} exactly once, transversally.

Remark

 ${\cal S}$ is a Poisson transversal for the KKS Poisson structure.

Definition

The universal centralizer of \mathfrak{g} is

$$\mathcal{Z} = \{(a, x) \in G \times \mathfrak{g} \mid x \in \mathcal{S}, a \in G^{\times}\}$$

$$\downarrow$$

$$\mathcal{S}.$$

 ${\mathcal Z}$ is a smooth, symplectic variety:

$$G \times G \subset T_G^* \cong G \times \mathfrak{g}$$

$$\downarrow^{\mu}$$

$$\mathfrak{g} \times \mathfrak{g}$$

Z is a smooth, symplectic variety:

$$G \times G \subset T_G^* \cong G \times \mathfrak{g} \quad (a,x)$$

$$\downarrow^{\mu} \qquad \downarrow^{\mu}$$

$$\mathfrak{g} \times \mathfrak{g} \quad (a \cdot x, x)$$

•
$$\mu^{-1}(x,x) = G^x \implies \mathcal{Z} = \mu^{-1}(\mathcal{S}_{\Delta}).$$

• the image of μ is $\{(x,y) \in \mathfrak{g} \times \mathfrak{g} \mid x \in G \cdot y\}$

$$\Rightarrow \mathcal{Z} = \mu^{-1}(\mathcal{S}_{\Delta}) = \mu^{-1}(\mathcal{S} \times \mathcal{S})$$

is a Poisson transversal in T_G^* .

G has a canonical smooth compactification \overline{G} , called the wonderful compactification.

Plan

Compactify the centralizer fibers of \mathcal{Z} in $\overline{\mathcal{G}}$.

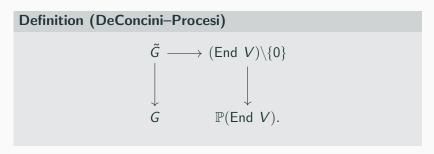
$$G \longrightarrow \overline{G}$$

$$T_G^* \longrightarrow T_{\overline{G},D}^*$$

Extend the symplectic structure on $\mathcal Z$ to a log-symplectic structure on its partial compactification.

The partial compactification of ${\mathcal Z}$

Let \tilde{G} be the simply-connected cover of G, V a regular irreducible \tilde{G} -representation.



The partial compactification of \mathcal{Z}

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Definition (DeConcini-Procesi)

The wonderful compactification of G is $\overline{G} := \overline{\chi(G)}$.

- independent of V
- smooth projective $G \times G$ -variety
- $D := \overline{G} \backslash G$ is a simple normal crossing divisor

The partial compactification of ${\cal Z}$

Example

Let $G = PGL_2 \quad \leadsto \quad \tilde{G} = SL_2, \quad V = \mathbb{C}^2$. Then

$$\chi(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2\times 2}) \mid ad - bc \neq 0 \right\},$$

and $\overline{G} = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3$.

$$D = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{P}(M_{2 \times 2}) \mid ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Non-example

Let $G=PGL_n$ for $n\geqslant 3$. Then $V=\mathbb{C}^n$ is not a regular rep of $\tilde{G}=SL_n$, and

$$\overline{G} \not\cong \mathbb{P}^{n^2-1}$$

The partial compactification of $\mathcal Z$

$$D=D_1\cup\ldots\cup D_l.$$
 $G imes G$ -orbits on $\overline{G}\longleftrightarrow J\subset\{1,\ldots,l\}$, in the sense that $\overline{\mathcal{O}_J}=\bigcap D_j.$

For each $J \subset \{1,\ldots,l\}$: parabolic subgroups P_J and P_J^- common Levi $L_J := P_J \cap P_J^-$, corresponding Lie algebras $\mathfrak{p}_J,\mathfrak{p}_J^-,\mathfrak{l}_J$.

$$\overline{L_J/Z(L_J)} \stackrel{\longleftarrow}{\longrightarrow} \overline{\mathcal{O}_J} \\
\downarrow \\
G/P_J \times G/P_J^-$$

The partial compactification of \mathcal{Z}

The log-cotangent bundle $T^*_{\overline{G},D}$ of \overline{G} fits into a short exact sequence

$$T_{\overline{G},D}^* \longleftrightarrow \overline{G} \times \mathfrak{g} \times \mathfrak{g} \longrightarrow T_{\overline{G},D}.$$

 $\leadsto T^*_{\overline{G},D}$ is a Lie algebroid over \overline{G} with trivial anchor map.

The fibers of $T_{\overline{G},D}^*$ are subalgebras of $\mathfrak{g} \times \mathfrak{g}$:

for each $J \subset \{1, \dots, I\}$, there is a basepoint $z_J \in \mathcal{O}_J$ such that

$$T^*_{\overline{G},D,z_J}=\mathfrak{p}_J\times_{\mathfrak{l}_J}\mathfrak{p}_J^-.$$

The partial compactification of ${\mathcal Z}$

Definition

$$\overline{\mathcal{Z}} = \left\{ (a, x) \in \overline{G} \times \mathfrak{g} \mid x \in \mathcal{S}, a \in \overline{G^x} \right\}$$

$$\downarrow$$

$$\mathcal{S}.$$

- generic fiber is a smooth toric variety
- · special fibers are singular

The partial compactification of ${\mathcal Z}$

 $T^*_{\overline{G},D}$ has a natural log-symplectic Poisson structure, and

$$G \times G \subset T^*_{\overline{G},D} \xrightarrow{\overline{\mu}} \mathfrak{g} \times \mathfrak{g}.$$

- $\overline{\mu}$ is projection onto the fibers of $\overline{G} \times \mathfrak{g} \times \mathfrak{g}$.
- the image of $\overline{\mu}$ is $\mathfrak{g} \times_{\mathfrak{g}//G} \mathfrak{g}$.

$$\Rightarrow \overline{\mu}^{-1}(\mathcal{S}_{\Delta}) = \overline{\mu}^{-1}(\mathcal{S} \times \mathcal{S}) \subset T^*_{\overline{G},D}$$
 is a Poisson transversal.

Theorem (B.)

$$\overline{\mathcal{Z}} \cong \overline{\mu}^{-1}(\mathcal{S}_{\Delta}) \subset T^*_{\overline{G},D}$$

is a smooth, log-symplectic partial compactification of \mathcal{Z} .

Plan

Integrate this to a multiplicative picture:

$$\mathfrak{g} \leadsto \tilde{G}$$
.

 $G \subset \tilde{G}$ by conjugation

$$\tilde{G}^{r} = \{g \in \tilde{G} \mid \dim G^{g} = I\}.$$

Remark

This is the regular locus of the AKM quasi-Poisson structure on \tilde{G} , whose nondegenerate leaves are the conjugacy classes.

Theorem (Steinberg)

There is an I-dimensional affine subspace

$$\Sigma \subset \tilde{G}^r$$

which meets each regular conjugacy class in \tilde{G} exactly once, transversally.

Definition

The (multiplicative) universal centralizer of \tilde{G} is

$$\mathfrak{Z} = \left\{ (a,g) \in G \times \widetilde{G} \mid g \in \Sigma, a \in G^g \right\}$$

$$\downarrow$$

$$\Sigma.$$

The double $\mathbb{D}_G:=G imes \tilde{G}$ has a natural q-Poisson structure with group-valued moment map

$$\tilde{G} \times \tilde{G} \subset \mathbb{D}_G$$

$$\downarrow^{\mu}$$
 $\tilde{G} \times \tilde{G}$

The double $\mathbb{D}_G:=G imes ilde{G}$ has a natural q-Poisson structure with group-valued moment map

$$\tilde{G} \times \tilde{G} \subset \mathbb{D}_{G} \qquad (a,g)$$

$$\downarrow^{\mu} \qquad \downarrow^{\mu}$$

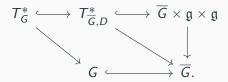
$$\tilde{G} \times \tilde{G} \quad (aga^{-1},g^{-1})$$

Proposition (Finkelberg-Tsymbaliuk)

$$\mathfrak{Z} = \mu^{-1}(\Sigma_{\Delta}) = \mu^{-1}(\Sigma \times \iota(\Sigma)) \subset \mathbb{D}_{G}$$

is a smooth, symplectic algebraic variety.

Recall the inclusions



Proposition (B.)

 $T^*_{\overline{G},D}$ integrates to a smooth subgroupoid

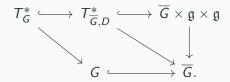
$$\mathbb{D}_{\overline{G}} \longleftrightarrow \overline{G} \times \widetilde{G} \times \widetilde{G}$$

$$\downarrow \downarrow$$

$$\overline{G}$$

whose source/target fiber at $z_I \in \overline{G}$ is $P_I \times_{L_I} P_I^-$.

Recall the inclusions



Proposition (B.)

 $\mathcal{T}^*_{\overline{G},D}$ integrates to a smooth subgroupoid

$$\mathbb{D}_{G} \longleftrightarrow \mathbb{D}_{\overline{G}} \longleftrightarrow \overline{G} \times \widetilde{G} \times \widetilde{G}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad \downarrow \qquad \qquad \qquad \qquad \qquad$$

whose source/target fiber at $z_l \in \overline{G}$ is $P_l \times_{L_l} P_l^-$.

Definition

$$\overline{\mathfrak{Z}} := \left\{ (a,g) \in \overline{G} \times \widetilde{G} \mid g \in \Sigma, a \in \overline{G^g} \right\}.$$

 $\mathbb{D}_{\overline{G}}$ has a Hamiltonian q-Poisson structure with moment map

$$G \times G \subset \mathbb{D}_{\overline{G}} \xrightarrow{\overline{\mu}} \tilde{G} \times \tilde{G}.$$

Theorem (B. in progress)

$$\overline{\mathfrak{Z}} \cong \overline{\mu}^{-1}(\Sigma_{\Delta}) = \overline{\mu}^{-1}(\Sigma \times \iota(\Sigma)) \subset \mathbb{D}_{\overline{G}}$$

is a smooth, log-symplectic partial compactification of \mathfrak{Z} .