A Singular Symplectic Slice Theorem

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Outline

Motivating examples

symplectic

b^m-Symplectic Manifolds

b-manifold: a pair (M, Z) of an oriented manifold M and an oriented exceptional hypersurface Z

defining function: $t : M \to \mathbb{R}, t|_Z = 0$

b-vector field: a vector field on M that is everywhere tangent to Z locally generated by $(t\frac{\partial}{\partial t}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$

b-tangent bundle: all the sections are b-vector fields, at $p \in M \setminus Z$, ${}^bT_pM = T_pM$

b-symplectic form: $\frac{dt}{t} \wedge \alpha + \beta$, $\alpha \in \Omega^1(M), \beta \in \Omega^2(M)$
symplectic at $p \in M \setminus Z$
symplectic at $p \in Z$ as an element of $\bigwedge^2({}^bT_p^*M)$

b^m-symplectic form: $\sum_{i=1}^{m} \frac{dt}{t^i} \wedge \alpha_i + \beta$

b^m-symplectic manifolds

$\Lambda^2({}^{b^m}T^*M)$

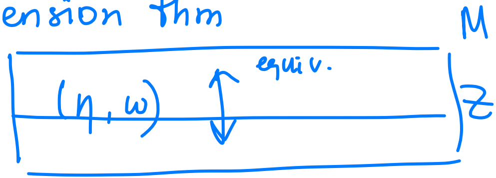
A b^m-form ω is called b^m-symplectic when it is closed and non-degenerate.
As a Poisson manifold, b^m-symplectic manifold admits induced symplectic foliation:

- The connected components of $M \setminus Z$ are open symplectic leaves of dimension $2n$
- Z admits a corank 1 Poisson (cosymplectic) structure

Definition

A cosymplectic structure on a manifold Z of an odd dimension $2n-1$ is a pair (η, ω), where η is a closed 1-form and ω is a closed 2-form such that $\eta \wedge \omega^{n-1}$ is a volume form on Z.

Extension thm

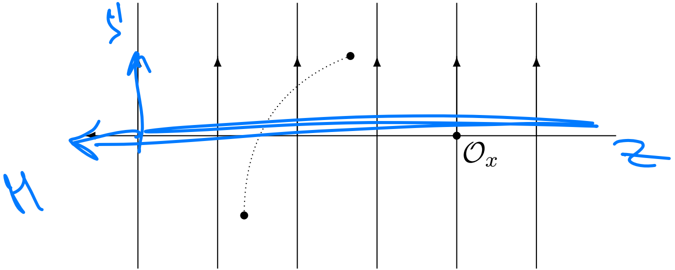


$$\underbrace{\pi^*\eta \wedge \frac{dt}{t} + \pi^*\omega}_{b\text{-symplectic}}$$

Group Actions

Theorem (Braddell, Kiesenhofer, Miranda)

Let G be a compact Lie group acting on a compact b^m-symplectic manifold. Then G is of the form $S^1 \times H \mod \mathbb{Z}_k$.



Hamiltonian Spaces

Definition

A Hamiltonian G-space (M, ω, μ) is a $2n$-dimensional manifold M with G-action, invariant 2-form $\omega \in \Omega^2(M)$ and an equivariant moment map $\mu : M \to \mathfrak{g}^*$ such that

(a1) ω is closed: $d\omega = 0$

(a2) moment map condition: $\iota(v_\xi)\omega = d\langle \phi, \xi \rangle, \forall \xi \in \mathfrak{g}$

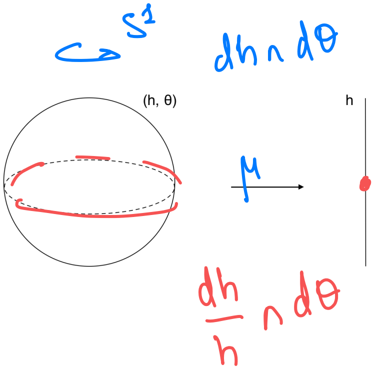
(a3) ω is non-degenerate

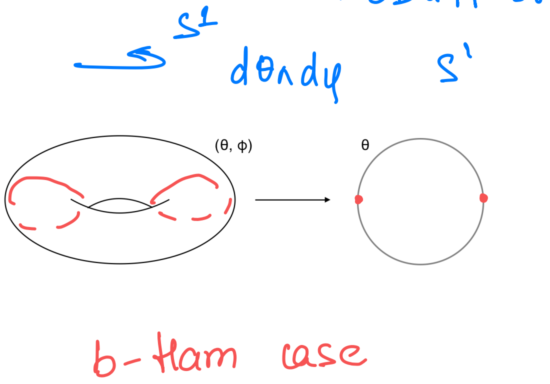
$\langle , \rangle$ – natural pairing identifying $\mathfrak{g}$ and $\mathfrak{g}^*$

v_ξ – generating vector field on M

Examples of actions under consideration

non-Ham McDuff '88





Here projection on h is a moment map for S^1-action

Projection on θ can not be taken as a moment map since $\theta \notin \mathfrak{g}^*$

b^m-Hamiltonian Spaces

Definition

The action of G on a b^m-symplectic manifold (M, Z, ω) is called b^m-Hamiltonian if there exists a moment map $\mu : M \to {}^{b^m}\mathcal{C}^\infty(M) \otimes \mathfrak{g}^*$ with

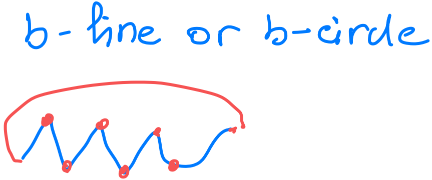
$$\iota(v_\xi)\omega = \langle d\mu, \xi \rangle$$

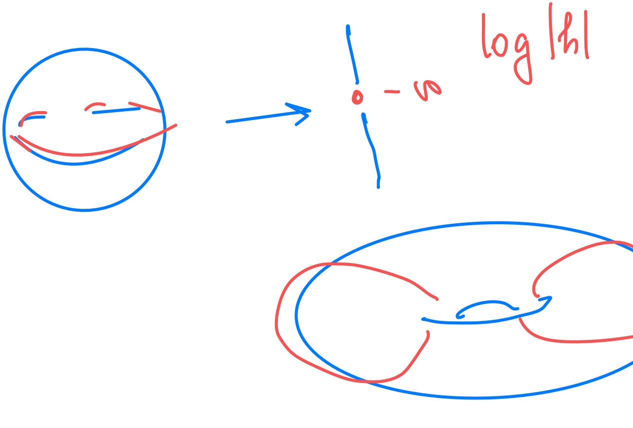
where ${}^{b^m}\mathcal{C}^\infty(M) = \left(\bigoplus_{i=1}^{m-1} t^{-i}\mathcal{C}^\infty(t) \right) \oplus {}^b\mathcal{C}^\infty(M)$ and ${}^b\mathcal{C}^\infty(M) = \{a \log|t| + g, g \in \mathcal{C}^\infty(M)\}$.

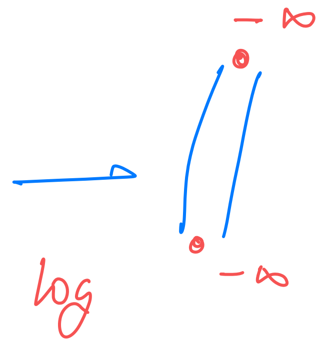
In other words, the action is b^m-Hamiltonian if it preserves b^m-symplectic form and ι_{v_ξ} is exact.

b-Moment Maps

b-line [Guillemin, Miranda, Pires, Scott] v1 (arXiv)







Superintegrable Systems

Definition

Let (M, Π) be a Poisson manifold of (maximal) rank $2r$. An s-tuple of functions $F = (f_1, \ldots, f_s)$ on M is a non-commutative integrable system of rank r on (M, Π) if

- $f_1, \ldots, f_s$ are independent (i.e. their differentials are independent on a dense open subset of M);
- The functions $f_1, \ldots, f_r$ are in involution with the functions $f_1, \ldots, f_s$;
- $r + s = \dim M$;
- The Hamiltonian vector fields of the functions $f_1, \ldots, f_r$ are linearly independent at some point of M.

Viewed as a map, $F : M \to \mathbb{R}^s$ is called the moment map of (M, Π, F).
When all the integrals commute, i.e. $r = s$, then we are dealing with the conventional case of a commutative integrable system.

Examples of Integrable Systems

Kepler problem

- Non-commutative integrable systems on manifolds with boundary
  (N, ω_N) – any symplectic manifold, H_+ – upper hemisphere with $\omega_H = \frac{1}{h} dh \wedge d\theta$.
  $(f_1, \ldots, f_s)$ – non-commutative integrable system of rank r on N.
  $(\log |h|, f_1, \ldots, f_s)$ – (smooth) non-commutative integrable system on the interior of $M = N \times H_+$
  On the double of M we have a non-commutative b-integrable system on $N \times S^2$.
- Examples coming from b-Hamiltonian $\mathbb{T}^r$-actions
- The geodesic flow
- The Galilean group

Slice Theorem

G – compact Lie group
W – abstract smooth manifold
$\mathcal{O}_x = \{y \in W | y = g \cdot x \text{ for some } g \in G\}$ – the orbit of x
$G_x = \{g \in G | g \cdot x = x\}$ – the stabilizer of x
f_x – orbit map

$$f_x : G \longrightarrow W$$

$$g \longmapsto g \cdot x$$

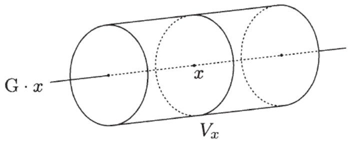
V_x – quotient vector space $T_xW/T\mathcal{O}_x$ (slice)

$$\begin{array}{ccc} G/G_x & \longrightarrow & G \times_{G_x} V_x \\ \downarrow f_x & & \downarrow \bar{f}_x \\ \mathcal{O}_x & \longrightarrow & W \end{array}$$

Slice Theorem

Theorem (Palais)

There exists an equivariant diffeomorphism from an equivariant open neighborhood of the zero section in $G \times_{G_x} V_x$ to an open neighborhood of $\mathcal{O}_x$ in W, which sends the zero section G/G_x onto the orbit $G \cdot x$ by the natural map f_x.



Symplectic (Hamiltonian) Slice Theorem

Theorem (Guillemin-Sternberg, Marle)

Let (M, ω, G) be a symplectic manifold together with a Hamiltonian group action. Let p be a point in M such that $\mathcal{O}_p$ is contained in the zero level set of the moment map. Denote G_p the stabilizer and $\mathcal{O}_p$ the orbit of p. There is a G-equivariant symplectomorphism from a neighbourhood of the zero section of the bundle $T^*G \times_{G_p} V_p$ equipped with symplectic model to a neighbourhood of the orbit $\mathcal{O}_p$.

Local Normal Form Theorem

Theorem (Guillemin-Sternberg)

Let (M, ω, μ) be a Hamiltonian G-space. For any $p \in M$, let $H = Stab(p)$, let $K = Stab(\mu(p))$, and let V be the symplectic slice at p. There exists a neighbourhood of the orbit $G \cdot p$ which is equivariantly diffeomorphic to a neighborhood of the orbit $G \cdot [e, 0, 0]$ in

$$Y := G \times H((\mathfrak{h}^0 \cap \mathfrak{k}^*) \times V).$$

In terms of this diffeomorphism, the moment map $\mu : M \to \mathfrak{g}^*$ may be written as

$$\mu([g, \gamma, v]) = Ad^*_g(\mu(p) + \gamma + \phi(v)),$$

where $\phi : V \to \mathfrak{h}$ is the moment map for the slice representation.

For $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, $\mathfrak{h}^0 \subset \mathfrak{g}^*$ denotes its annihilator.

b^m-Symplectic Slice Theorem

Theorem (b^m- Hamiltonian)

Let $S^1 \times H$ be a compact group acting on a b^m-symplectic manifold (M, Z, ω). Let $\mathcal{O}_z$ be an orbit of the group contained in the critical set of M. Then there is a neighbourhood of the zero section of an associated bundle ${}^{b^m}T^*(H \times S^1) \times_{H_z \times \mathbb{Z}_d} V_z$ equipped with the b^m-symplectic model

$$\omega = \sum c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$$

where t is a defining function for Z, π is the projection $\pi : T^*S^1 \times T^*H \times_{H_z} V_z \to T^*H \times_{H_z} V_z$ and ω_H is the symplectic form on $T^*H \times_{H_z} V_z$ given by the symplectic slice theorem.

The moment map is given by

Hamiltonian

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_i \frac{t^{-i}}{i} + \mu_0(x, y)$$

Description of the Model

- SST: there is an H-equivariant neighbourhood U_H of $\mathcal{O}_p^H$ which is equivariantly symplectomorphic to $T^*H \times_{H_p} V_p$ with the symplectic form ω_H on $T^*H \times_{H_p} V_p$.
- Consider

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$$

where $\pi : T^*S^1 \times T^*H \times_{H_p} V_p \to T^*H \times_{H_p} V_p$.

- Consider the quotient b^m-Poisson structure on $T^*(S^1 \times H) \times_{H_p \times \mathbb{Z}_d} V_p$ where $\mathbb{Z}_d$ acts on T^*S^1 as the cotangent lift of $\mathbb{Z}_d$ acting by translations on S^1 and by linear symplectomorphisms on V_p and H_p acts on T^*H by the cotangent lift of H_p acting on H by translations and by linear symplectomorphisms on V_p.
- This is b^m-symplectic model on the associated vector bundle $T^*(S^1 \times H) \times_{(H_p \times \mathbb{Z}_d)} V_p$.

b^m-cotangent lift

Given an action ρ of a Lie group G on a smooth manifold M, one can lift it to the (b^m-)Hamiltonian action $\hat{\rho}$ of G on the cotangent bundle T^*M. $\hat{\rho}$ is given by $\hat{\rho}_g := \rho_{g^{-1}}$ and π is a canonical projection from T^*M to M. The following diagram commutes:

$$\begin{array}{ccc} T^*M & \xrightarrow{\hat{\rho}_g} & T^*M \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\rho_g} & M \end{array}$$

b^m-cotangent lift II

Having the action $S^1 \times H \curvearrowright T^*\mathbb{S}^1 \times T^*H$ we consider the coordinates $(a, \theta, x_1, \dots, x_n, y_1, \dots, y_n)$ with $\theta \in S^1, \{x_i\} \in H$ and $a, \{y_i\} \in \mathbb{R}$. Here H is itself an $(n-1)$-dimensional manifold and T^*H is equipped with standard Liouville one-form λ_H.

$$L = \sum_{1}^{m-1} c_i \frac{d\theta}{a^i} + c_0 \log a d\theta + \sum_{1}^{n-1} y_j dx_j$$

The action of $S^1 \times H$ on its cotangent bundle is Hamiltonian with the moment map given by contraction of λ with the fundamental vector field:

$$\langle \mu(p), X \rangle := \left\langle L_p, X^{\#}|_p \right\rangle$$

b^m-cotangent lift III

We should prove that the Liouville form is invariant under this action. L splits in two: λ_H and λ. One has two show that λ_H is invariant under S^1-action and for λ we already have it proven from the standard symplectic cotangent lift.
The moment map then is given by

$$\mu = c_1 \log|a| + \sum_{i=1}^{m-1} c_i \frac{a^{-i}}{i} + \mu_0(x,y)$$

$$\tilde{\omega} = \sum_{0}^{m-1} \frac{c_i}{a_1^{i+1}} d\theta_1 \wedge da_1 + \sum_{2}^{n} dx_j \wedge dy_j$$

Desingularization

Theorem (Guillemin-Miranda-Weitsman)

Let ω be a b^m-symplectic structure on a compact manifold M and let Z be its critical hypersurface.

- *If m is even, there exists a family of symplectic forms ω_ε which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bi-vector fields $(\omega_\varepsilon)^{-1}$ converges in the $\mathcal{C}^{2m-1}$-topology to the Poisson structure ω^{-1} as $\varepsilon \to 0$.*
- *If m is odd, there exists a family of folded symplectic forms ω_ε which coincide with the b^m-symplectic form ω outside an ε-neighborhood of Z.*

Definition

The pair $(M^{2n}, \omega \in \Omega^2(M))$ is called a folded symplectic manifold if the top power ω^n vanishes transversally on a folding hypersurface Z and its restriction to that submanifold has maximal rank.

Marsden-Weinstein Reduction

Theorem (Hamiltonian reduction)

Let (M, ω, μ) be a Hamiltonian G-space for a compact Lie group G. Let $i : \varphi^{-1}(0) \hookrightarrow M$ be the inclusion map. Assume that G acts freely on $\mu^{-1}(0)$. Then

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a manifold,*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle,*
- *there is a symplectic form ω_{red} on M_{red} satisfying $i^*\omega = \pi^*\omega_{red}$.*

$$
\begin{array}{ccccc}
(M, \omega, Z) & \xrightarrow{slice} & (V_p, \omega_p, Z) & \xrightarrow{red} & (M//G, \omega_{red}) \\
\downarrow & & \downarrow & & \uparrow \\
(M, \omega_\varepsilon, Z) & \xrightarrow{slice} & (V_p, \omega_{\varepsilon p}, Z) & \xrightarrow{red} & (M//G, \omega_{\varepsilon red})
\end{array}
$$

What if the action is non-Hamiltonian?

[Ortega-Ratiu'01] Symplectic Slice Theorem
arXiv:math/0110084

Consider symplectic actions that are tubewise Hamiltonian, chu map and cylinder-valued moment maps. Only works for abelian case. Easily extends previous result for the singular Hamiltonian actions.

[Bott-Tolman-Weitsman'02] quasi-Hamiltonian Slice Theorem
arXiv:math/0210036

Consider quasi-Hamiltonoan spaces. By cross-section theorem show that action on the slice corresponds to some associated Hamiltonian space and brings it back to the q-Ham normal form theorem.

quasi-Hamiltonian Spaces

Definition

A quasi-Hamiltonian G-space is a $2n$-dimensional manifold M with G-action, invariant 2-form σ and equivariant moment map $\Phi : M \to G$ such that:

(b1) σ is equivariantly closed: $d\sigma = -\Phi^*\chi$

(b2) moment map condition: $\iota(v_\xi)\sigma = \frac{1}{2}\Phi^*\left\langle\theta^l + \theta^r, \xi\right\rangle$

(b3) σ is weakly non-degenerate

where θ^l and θ^r are left- and right-invariant Maurer-Cartan forms and $\chi \in \Omega^3_G(G)$ is canonical closed bi-invariant 3-form.
In matrix representation, $\theta^l = g^{-1}dg$, $\theta^r = dgg^{-1}$ and $\chi = \frac{1}{12}(\theta, [\theta, \theta])$

Examples of quasi-Hamiltonian Spaces

- Hamiltonian G-spaces
- Hamiltonian LG-spaces
- Conjugacy classes $\mathcal{C} \subset G$.
- Space of flat connections $\mathcal{A}(\Sigma)$ on a manifold with boundary $\partial\Sigma = S^1$ reduced with respect to the action of normal subgroup of gauge group $\mathcal{G}(\Sigma, \partial\Sigma) = \{\gamma \in \mathcal{G}(\Sigma) | \gamma|_{\partial\Sigma} = e\}$

From Hamiltonian to quasi-Hamiltonian

$\exp_s : \mathfrak{g} \to G$, $\exp_s(\eta) = \exp(s\eta)$

We take a new form $\breve{\omega} \in \Omega^2(\mathfrak{g})$

$$\breve{\omega} = \frac{1}{2}\int_0^1 (\exp_s^* \bar{\theta}, \frac{\partial}{\partial s}\exp_s^* \bar{\theta})ds$$

$\breve{\omega}$ is G-invariant and satisfies $d\breve{\omega} = -\exp^* \chi$.
$\mu = \exp \phi$
$\omega := \sigma + \breve{\omega}$

(M, ω, μ) is a quasi-Hamiltonian space

quasi-Hamiltonian Local Normal Form Theorem

Theorem (Bott-Tolman-Weitsman)

Let (M, σ, Φ) *be a quasi-Hamiltonian* G*-space. For any* $p \in M$*, let* $H = Stab(p)$*,* $K = Stab(\Phi(p))$*, and* V *be the symplectic slice at* p*. There exists a neighbourhood of the orbit* $\mathcal{O}_p$ *which is equivariantly diffeomorphic to a neighborhood of the orbit* $G \cdot [e, 0, 0]$ *in*

$$Y := G \times_H ((\mathfrak{h}^\perp \cap \mathfrak{k}) \times V).$$

In terms of this diffeomorphism, the G*-valued moment map* $\Phi : M \to G$ *may be written as* $\Phi([g, \gamma, v]) = Ad_g(\Phi(p) \exp(\gamma + \phi(v)))$*, where* $\phi : V \to \mathfrak{h}^* \simeq \mathfrak{h}$ *is the moment map for the slice representation.*

quasi-Hamiltonian Local Normal Form Theorem

Let (M, σ, Φ) be a quasi-Hamiltonian G-space. Let $U \subset \mathfrak{g}$ be a connected neighborhood of 0 so that the exponential map is a diffeomorphism on U, and let $V = \exp U$. Then there exists a Hamiltonian G-space (N, ω, ν) and an equivariant diffeomorphism $\psi : N \to \Phi^{-1}(V)$, so that the following diagram commutes

$$\begin{array}{ccc} N & \xrightarrow{\nu} & \mathfrak{g}^* \simeq \mathfrak{g} \\ \downarrow{\psi} & & \downarrow{\exp} \\ \Phi^{-1}(V) & \xrightarrow{\Phi|_{\Phi}^{-1}(V)} & g \end{array}$$

Sketch of proof:

$\nu := \log \Phi$ satisfies the moment map condition

$\omega = \sigma - \Phi^* \log^* \breve{\omega}$ is closed and non-degenerate

quasi-Hamiltonian Local Normal Form Theorem III

Theorem (Cross-Section)

Let (M, σ, Φ) be a quasi-Hamiltonian G-space. Given $g \in G$, let V_g be a slice for the action of G on itself at g, and let $Y_g := \Phi^{-1}(V_g)$. Let $K = Z(g)$ be the centralizer of g. The quasi-Hamiltonian cross-section $(Y_g, \sigma|_{Y_g}, \Phi|_{Y_g})$ is a quasi-Hamiltonian K-space.

Sketch of proof of the normal form theorem:
Consider $p \in M$, $g := \Phi(p)$, V_g – slice for G-action at p.
$(Y_g, \sigma|_{Y_g}, \Phi|_{Y_g})$ is a q-Ham $Z(g)$ space (by C-S thm)
Define $\psi : Y_g \to K$ as $\psi(m) = g^{-1}\Phi(m)$
$(Y_g, \sigma|_{Y_g}, \psi)$ is a q-Ham $Z(g)$ space, $\psi(p) = e$
Now consider (N, ω, ν), slice at p in N coincides with slice at p in Y_ζ so we can pass to symplectic slice theorem.

quasi-Hamiltonian Local Normal Form Theorem IV

Applying slice theorem to (N, ω, ν).
N is locally diffeomorphic to $Z(g) \times_H ((\mathfrak{h}^0 \cap \mathfrak{k}^* \times V)$
The diffeomorphism sends the moment map ν to the map

$$\nu \to ([k, \alpha, v] \to k \cdot (\alpha + \phi(v))),$$

where ϕ is the moment map for the slice representation.
Last step is to exponentiate this map.

$S^1 \times H$

$\mu : M \to {}^{b^m}C^\infty(M) \otimes G$

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Thank you for your attention!

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