Characteristic classes for Lie groupoids: going to the basics

Friday Fish, 24/07/2020

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Motivação $\quad M$-fixed

Characteristic classes: $P \to M$ ppal bdle, VB $\rightsquigarrow c^k(P) \in H^k(M)$

$\to$ **Chern classes:** $\quad E \to M \quad$ cplx V.B

②

Motivação $\quad M$-fixed

→ <u>Chern classes</u> : $E^n \to M$ cplx V.B $\rightsquigarrow c^k(E) \in H^k(M)$

Classifying space B

$$\begin{tikzcd} f^*Q^{univ} \simeq E \arrow[d] & Q^{univ} \arrow[d] \\ M \arrow[r, dashed, "f"] & B \end{tikzcd}$$

$c^i(E) := f^* c^i, \quad c^i \in H^i(B)$

Chern-Weil theory

Choose ∇: $\quad \nabla_X e^i = \omega^i_j(X) e^j$

$\rightsquigarrow$ curvature $\Omega : \Lambda^2 TM \to End(E)$

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega]$$

$$\det\left(\frac{i}{2\pi} t\Omega + I\right) = \sum_{k}^{n} c_k(E) t^k$$

$$c^k(E) \in H^{2k}(M)$$

③

Motivation:

Characteristic classes for $E \in \mathrm{Rep}(A)$: (Marius/Rui)

→ $c^k(E) \in H^{2k-1}(A)$: Image $VE : H_{dif}(G) \to H(A)$

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Motivation:

Characteristic classes for $E \in \mathrm{Rep}(A)$: (Marius/Rui)

→ **Construction**: $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$

Local frame e of $E_{|U}$: $\nabla_\alpha e^i = \omega^i_j(\alpha) e^j$, $\quad \alpha \in \Gamma(A)$

$$\omega_e := (\omega^i_j) \in C^1(A) \otimes \mathfrak{gl}_n \rightsquigarrow c_{2k-1}(E_{|U}) := \mathrm{tr}(\underbrace{\omega_e \wedge \dots \wedge \omega_e}_{2k-1}) \in H^{2k-1}(A)$$

$$\mathrm{tr}(\omega_e) - \mathrm{tr}(\omega_f) = d\xi \ \checkmark \quad \overset{\mathbb{R}}{\leftsquigarrow} E \overset{\mathbb{C}}{\rightsquigarrow} \quad \mathrm{tr}(\omega_e) - \mathrm{tr}(\omega_f) \neq d\xi \ \times$$

Replace ω_e by $\dfrac{\omega_e + \omega_e^*}{2}$ (???)

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Definition:

$$\begin{array}{c} \mathcal{G} \\ \downarrow\downarrow \\ M \end{array}, \quad \begin{array}{c} E \\ \downarrow \\ M \end{array} \in \mathrm{Rep}_{\mathbb{C}}(\mathcal{G}) : \quad \varphi_g : E_x \xrightarrow{\sim} E_y \quad \left(x \xrightarrow{g} y\right) \iff \begin{tikzcd} \mathcal{G} \arrow[r, "\varphi"] \arrow[dr, shift left] \arrow[dr, shift right] & GL(E) \arrow[d, shift left] \arrow[d, shift right] \\ & M \end{tikzcd}$$

Defn: $$H_{\mathbb{R}}(GL_n(\mathbb{C})) \xrightarrow[\sim]{ME} H(GL(E)) \xrightarrow{\varphi^*} H(\mathcal{G})$$

⑥

Definition of cochains

$G \xrightarrow{\varphi} GL(E)$ **Defn**: $H_{\mathbb{R}}(Gl_n(\mathbb{C})) \underset{\cong}{\xrightarrow{ME}} H(GL(E)) \xrightarrow{\varphi^*} H(G)$

Locally: $E = M \times \mathbb{C}^n \rightsquigarrow GL(E) \xrightarrow{r} Gl_n = Gl_n(\mathbb{C})$, $\quad r \circ F = \mathrm{id}$

$$\begin{array}{ccc} \xi : E_x \to E_y & \mapsto & A_\xi \\ \downarrow\downarrow & & \downarrow\downarrow \\ M & \longrightarrow & * \end{array}$$

$$C^\cdot(Gl_n) \xrightarrow{r^*} C^\cdot(GL(E)) \xrightarrow{\varphi^*} C^\cdot(G)$$

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→ G-Lie group : Cohomology of BG and

$R_{G/K} : W(\mathfrak{g}, \mathfrak{k}) \to \Omega^{\bullet}(B.G)$

→ Computation of $C^{\bullet}(\mathfrak{gl}_n, U(n)) \xrightarrow{R_{Gl_n/U(n)}} C^{\bullet}(Gl_n)$

(Polar decomposition of matrices)

$\rightsquigarrow$ local formulas for ch. classes.

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Cohomology of BG

$$\begin{array}{c} P \curvearrowleft G \\ \downarrow \\ M \end{array} \quad \text{ppal } G\text{-bdle} \quad \longleftrightarrow \quad \begin{tikzcd} & EG \curvearrowleft G \arrow[d, dashed] \\ M \arrow[r, "f"'] & BG \end{tikzcd}$$

Ch. classes P:

$f^* H^\bullet(BG)$

Model for $H^\bullet(BG)$: Bott-Shulman complex

$E_\bullet G$ - simplicial:

$$\begin{tikzcd} \vdots & \vdots & \vdots & \\ \Omega^2(E_0G) \arrow[r] \arrow[u, dashed] & \Omega^2(E_1G) \arrow[r] \arrow[u, dashed] & \Omega^2(E_2G) \arrow[r, dashed] \arrow[u, dashed] & {} \\ \Omega^1(E_0G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(E_1G) \arrow[r, "\delta"] \arrow[u] & \Omega^1(E_2G) \arrow[r, dashed] \arrow[u] & {} \\ \Omega^0(E_0G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(E_1G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(E_2G) \arrow[r, dashed] \arrow[u, "d"] & {} \end{tikzcd}$$

$$E_{p+1}G = G^{p+2} \xrightarrow{\partial^i} E_pG = G^{p+1}$$

Cohomology of BG

Model for $H^\bullet(BG)$: Bott-Shulman complex $E_\bullet G$-simplicial:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
\Omega^2(E_0G) \arrow[r] \arrow[u, dashed] & \Omega^2(E_1G) \arrow[r] \arrow[u, dashed] & \Omega^2(E_2G) \arrow[r, dashed] \arrow[u, dashed] & {} \\
\Omega^1(E_0G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^1(E_1G) \arrow[r, "\delta"] \arrow[u] & \Omega^1(E_2G) \arrow[r, dashed] \arrow[u] & {} \\
\Omega^0(E_0G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(E_1G) \arrow[r, "\delta"] \arrow[u, "d"] & \Omega^0(E_2G) \arrow[r, dashed] \arrow[u, "d"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
E_pG = G^{p+1} & G \arrow[l, bend right] \\
B_pG = G^p \arrow[from=1-1]
\end{tikzcd}$$

$\rightsquigarrow x \in \mathfrak{g}$: i_x, L_x

$i_x : \Omega^q(E_pG) \to \Omega^{q-1}(E_pG)$

$L_x : \Omega^q(E_pG) \to \Omega^q(E_pG)$

→ $\Omega^\bullet(E_\bullet G)$ - $\mathfrak{g}$-dg-algebra

→ $\Omega^\bullet(E_\bullet G)_{basic} = \operatorname{Ker} i_x \cap \operatorname{Ker} L_x$

$$H(\Omega^\bullet(E_\bullet G)_{basic}) = H(BG)$$

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Chern-Weil theory

$\mathfrak{g}$-Lie algebra $\rightsquigarrow$ **Weil algebra:**

$$W^{p,q} = S^p \mathfrak{g}^* \otimes \Lambda^{p-q} \mathfrak{g}^* \qquad p,q \geq 0$$

$\to$ $\mathfrak{g}$-dg-algebra: i_X - contraction on $\Lambda \mathfrak{g}^*$, trivial on $S\mathfrak{g}^*$

$L_X = ad^*_X$, $\qquad X \in \mathfrak{g}$

$\mathfrak{g} = Lie(G)$ $\qquad \mathfrak{g}^* \xrightarrow{\theta} \Omega^1(G) = \Omega^1(E_0 G)$ – connection

left-inv. MC

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Chern-Weil theory

Weil algebra: $W^{p,q} = S^p\mathfrak{g}^* \otimes \Lambda^{p-q}\mathfrak{g}^*$

$\mathfrak{g}$-dg-algebra: $i_x : W^\bullet \to W^{\bullet-1}$, $L_x : W^\bullet \to W^\bullet$ $\quad x \in \mathfrak{g}$

$\mathfrak{g} = \mathrm{Lie}(G)$ $\qquad \mathfrak{g}^* \xrightarrow{\theta} \Omega^1(G) = \Omega^1(E.G)$ – connection

Thm (Alexeev-Meinrenken) $W\mathfrak{g} \xrightarrow{c^\theta} \Omega^\bullet(E.G)$ $\quad$ $\mathfrak{g}$-dg-spaces

$(S\mathfrak{g}^*)_{inv} \xrightarrow{c^\theta} \Omega^\bullet(B.G)$ $\quad$ $\mathfrak{g}$-dg-algebras in cohom.

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Chern-Weil theory

$\mathfrak{g} = \mathrm{Lie}(G)$: $\quad \mathfrak{g}^* \xrightarrow{\theta} \Omega^1(G) \xrightarrow{(A\text{-}M)} W\mathfrak{g} \xrightarrow{c^\theta} \Omega^\cdot(E.G)$ $\quad \mathfrak{g}$-dg-sp.

Thm (Alexeev-Meinrenken) G-compact, connected

$\mathfrak{g}$-dg-algebras in cohom. $\quad (S\mathfrak{g}^*)_{inv} \xrightarrow{c^\theta} \Omega^\cdot(B.G)$ is a h. equiv.

Thm (S.) G-connected, $K \subset G$ max. compact subgroup with $\mathfrak{k} := \mathrm{Lie}(K)$. Then

$$(W\mathfrak{g})_{\mathfrak{k}\text{-basic}} \xrightarrow{c^\theta} \Omega^\cdot(E.G)_{\mathfrak{k}\text{-basic}} \xrightarrow{\bar{\imath}^*} \Omega^\cdot(B.G)$$ is a h. equiv

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Chern-Weil theory

$\mathfrak{g} = \mathrm{Lie}(G)$: $\mathfrak{g}^* \xrightarrow{\theta} \Omega^1(G) \rightsquigarrow W\mathfrak{g} \xrightarrow{c^\theta} \Omega^\bullet(E.G)$

Thm (S.) G-connected, $K \subset G$ max. compact, $\mathfrak{k} = \mathrm{Lie}(K)$

$$(W\mathfrak{g})_{\mathfrak{k}\text{-basic}} \xrightarrow{c^\theta} \Omega^\bullet(E.G)_{\mathfrak{k}\text{-basic}} \xrightarrow{\overline{\pi^*}} \Omega^\bullet(B.G) \quad \text{is a h. equiv}$$

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Chern-Weil theory

Thm (S.) G-connected, $K \subset G$ max. compact, $\mathfrak{k} = \mathrm{Lie}(K)$

1. $(W\mathfrak{g})_{\mathfrak{k}\text{-basic}} \xrightarrow{c^\theta} \Omega^\bullet(E_\bullet G)_{\mathfrak{k}\text{-basic}} \xrightarrow{i^*} \Omega^\bullet(B_\bullet G)$ is a h. equiv

2. $R_{G/K} : (\Lambda^\bullet \mathfrak{g}^*)_{\mathfrak{k}\text{-basic}} \xrightarrow{c^\theta} \Omega^\bullet(E_\bullet G)_{\mathfrak{k}\text{-basic}} \xrightarrow{i^*} \Omega^\bullet(B_\bullet G) = C^\bullet(G)$

Van Est Integration (Meinrenken-S.)

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Description of $H_{dR}(GL_n\mathbb{C})$ at cochains

$U(n) \subset GL_n$ max. compact, $\mathrm{Lie}\, U(n) = \mathfrak{u}(n) = \{A \in \mathfrak{gl}_n \mid A^* = -A\}$

$$(\Lambda^\bullet \mathfrak{gl}_n^*)_{\mathfrak{u}(n)\text{-basic}} \xrightarrow{R_{GL_n/U(n)}} C^\bullet(GL_n)$$

Polar decomposition

$\mathfrak{p} = \{A \in \mathfrak{gl}_n \mid A = A^*\}$

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p} \quad : \quad A = \frac{A - A^*}{2} + \frac{A + A^*}{2}$$

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Description of $H_{dif}(GL_n\mathbb{C})$ at cochains

Polar decomp: $\mathfrak{u}(n)=\{A\in\mathfrak{gl}_n \mid A^*=-A\}$, $\mathfrak{p}=\{A\in\mathfrak{gl}_n \mid A=A^*\}$

$$\mathfrak{gl}_n(\mathbb{C})=\mathfrak{u}(n)\oplus\mathfrak{p} \quad : \quad A=\frac{A-A^*}{2}+\frac{A+A^*}{2}$$

a. $[\mathfrak{u}(n),\mathfrak{u}(n)]\subset\mathfrak{u}(n)$, b. $[\mathfrak{u}(n),\mathfrak{p}]\subset\mathfrak{p}$, c. $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{u}(n)$

Lemma: $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$. If b. $\Rightarrow (\Lambda^\bullet\mathfrak{g}^*)_{\mathfrak{k}\text{-basic}}=(\Lambda^\bullet\mathfrak{p}^*)_{\mathfrak{k}\text{-inv}}$

If b & c $\Rightarrow H^\bullet(\mathfrak{g},\mathfrak{k})=(\Lambda^\bullet\mathfrak{p}^*)_{\mathfrak{k}\text{-inv}}$

Description of $H_{dR}(GL_n\mathbb{C})$ at cochains

$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p}$: $\quad H^\bullet(\mathfrak{gl}_n, \mathfrak{u}(n)) = (\Lambda^\bullet \mathfrak{p}^*)_{\mathfrak{u}(n)-\mathrm{inv}}$

Characteristic coeff. $c_q \in (S^q \mathfrak{gl}_n^*)_{\mathrm{inv}}$: $\det(tI + A) = \sum_{q=0}^{n} c_q t^{n-q}$

$$(S\mathfrak{gl}_n^*)_{\mathrm{inv}} = \mathbb{C}[c_1, \dots, c_n] \qquad c_q(A) = \mathrm{tr}(\Lambda^q A)$$

→ Cartan map: $\sigma : (S^q \mathfrak{gl}_n^*)_{\mathrm{inv}} \longrightarrow (\Lambda^{2q-1} \mathfrak{gl}_n^*)_{\mathrm{inv}}$

$$\sigma(c_q) = \mathrm{cte}\, \Phi_{2q-1}, \quad \Phi_{2q-1}(A_1, \dots, A_{2q-1}) = \sum_{\gamma \in S_{2q-1}} \varepsilon^\gamma \mathrm{tr}(A_{\gamma(1)} \cdots A_{\gamma(2q-1)})$$

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Description of $H_{dR}(GL_n\mathbb{C})$ at cochains

$$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{p} : \quad H(\mathfrak{gl}_n, \mathfrak{u}(n)) = (\Lambda^\bullet \mathfrak{p}^*)_{\mathfrak{u}(n)\text{-inv}} = (\Lambda^\bullet \mathfrak{gl}_n^*)_{\mathfrak{u}(n)\text{-basic}}$$

$A^* = -A \qquad A^* = A$

$\rightarrow$ Cartan map: $\sigma : (S^q \mathfrak{gl}_n^*)_{inv} \longrightarrow (\Lambda^{2q-1} \mathfrak{gl}_n^*)_{inv}$

$$\sigma(C_q) = cte\, \Phi_{2q-1}, \quad \Phi_{2q-1}(A_1, \dots, A_{2q-1}) = \sum_{\gamma \in S_{2q-1}} \varepsilon^\gamma tr(A_{\gamma(1)} \cdots A_{\gamma(2q-1)})$$

$$\langle u_{2q-1} := i^{q-1} \Phi_{2q-1} | \mathfrak{p} \rangle_{\mathbb{R}} = (\Lambda^{2q-1} \mathfrak{p}^*)_{\mathfrak{u}(n)\text{-inv}} = H^{2q-1}(\mathfrak{gl}_n, \mathfrak{u}(n))$$

$$H(\mathfrak{gl}_n, \mathfrak{u}(n)) = \Lambda(u_1, u_3, \dots, u_{2n-1})$$

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Description of $H_{dif}(GL_n\mathbb{C})$ at cochains

$$(\Lambda^\bullet \mathfrak{gl}(n^*))_{U(n)\text{-basic}} \xrightarrow{R_{GL_n/U(n)}} C^\bullet(GL_n): \quad H(GL_n) = \Lambda(v_1, v_3, \dots, v_{2n-1})$$

$$u_{2q-1} \longmapsto v_{2q-1}$$

Polar decomposition $\quad GL_n = P\,U(n)$; $\quad e: \mathfrak{p} \xrightarrow{\simeq} P$

$A = e^X U: \ X \in \mathfrak{p}, \ U \in U(n)$

$$GL_n/U(n) = P \overset{e}{\cong} \mathfrak{p}$$

$$e^X U(n) \leftrightarrow e^X \leftrightarrow X$$

$$\rightsquigarrow t \cdot e^X U(n) := e^{tX} U(n), \qquad t \in \mathbb{R}$$

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Description of $H_{dR}(GL_n\mathbb{C})$ at cochains

$$(\Lambda^\cdot \mathfrak{gl}_n^*)_{U(n)\text{-basic}} \xrightarrow{R_{GL_n/U(n)}} C^\cdot(GL_n):\quad H(GL_n)=\Lambda(v_1,v_3,\dots,v_{2n-1})$$

$$(\Lambda^\cdot \mathfrak{p}^*)_{U(n)\text{-inv}} \qquad u_{2q-1} \mapsto v_{2q-1}$$

$\to \mathfrak{gl}_n = \mathfrak{p}\oplus \mathfrak{u}(n)$

$\to GL_n = P\,U(n)$

$$v_1 = \underline{R_{GL_n/U(n)}(tr_{|\mathfrak{p}})} \in C^1(GL_n):\ (\Lambda^1\mathfrak{p})_{U(n)\text{-inv}} \simeq \Omega^1(P)^{GL_n},\quad \alpha\mapsto\alpha_P$$

$$A=e^X U:\quad t \overset{\gamma_A(t)}{\longmapsto} e^{tX}\in P$$

$$v_1(e^X U)=\int_0^1 \gamma_A^*(tr_P)\left(\frac{\partial}{\partial t}\right)=\int tr\left(L_{e^{-tX}}\left(\frac{d e^{sX}}{ds}\Big|_{s=t}\right)\right)=tr(X)$$

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The 1-st ch. class $V_1(E) \in C^1(\mathcal{G})$

$\mathcal{G} \xrightarrow{\varphi} GL(E)$ **Defn**: $H_{\mathbb{R}}(Gl_n(\mathbb{C})) \underset{\simeq}{\xrightarrow{ME}} H(GL(E)) \xrightarrow{\varphi^*} H(\mathcal{G})$

Locally: $E = M \times \mathbb{C}^n \rightsquigarrow GL(E) \xrightarrow{r} Gl_n = Gl_n(\mathbb{C})$,

$\xi : E_x \to E_y \mapsto A_\xi$

$$(\Lambda^{\cdot}\mathfrak{gl}_n^*)_{U(n)\text{-b}} \overset{R}{\dashrightarrow} C^{\cdot}(Gl_n) \xrightarrow{r^*} C^{\cdot}(GL(E)) \xrightarrow{\varphi^*} C^{\cdot}(\mathcal{G})$$

$V_1(E)(g) = tr(X_g)$

$A_g = e^{X_g} U_g : E_{s(g)} \to E_{t(g)}$, $X_g \in \mathfrak{p}$

VE $\rightsquigarrow$ $U_1(E)(\alpha) = tr\left(\frac{\omega(\alpha) + \omega^*(\alpha)}{2}\right)$

$U_1 \in C^1(A)$

$\omega = \left(\frac{\omega + \omega^*}{2}\right) + \left(\frac{\omega - \omega^*}{2}\right)$, $\frac{\omega+\omega^*}{2} \in \mathfrak{p}$

Questions:

- → Ch. classes for rep. up to homotopy
  $\rightsquigarrow$ intrinsic classes for groupoids (Adjoint)
- → Applications