On singular cotangent homotopies coming from the Poisson Sigma Model

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Plan:

Intro: the semiclassical limit in the PSM and quantization
Our singular homotopies and groupoid triangles
Conclussions and outlook

Introduction

Given a Poisson manifold (M, π)

assoc.

$\hbar \mapsto \star_\hbar \quad C^\infty(M)[[\hbar]]$

$f_1 \star_\hbar f_2 = f_1 \cdot f_2 + \frac{\hbar}{2}\{f_1, f_2\}_\pi + O(\hbar^2)$

quantization by star products $\xrightarrow[\lim\ (\hbar \to 0)]{\text{Semiclassical}}$ (local/formal) symplectic groupoids

$M \simeq \mathbb{R}^n$

$p : \mathbb{R}^n \to \mathbb{R}$ lin

$\left(e^{\frac{i}{\hbar}p_1} \star_\hbar e^{\frac{i}{\hbar}p_2}\right)(x)$

$= \underbrace{a_\hbar(p_1, p_2, x)}_{\text{reg } \hbar \to 0} \, e^{\frac{i}{\hbar} S_P(p_1, p_2, x)}$

"SC lim"

[Kont] → '97

[Catt, Feld]

quantum PSM (on disk) $\xrightarrow{\text{SC lim}}$ (local/formal) symplectic groupoids

$\star_\hbar$ assoc $\Rightarrow$ S_P generates local sympl. gd

st. $G_P \subseteq T^*M$

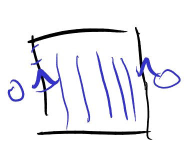
$\downdownarrows$

(M, π)

Rmk: classical hamiltonian picture of PSM on IxI and the Weinstein groupoid

integr. (M, π)

[Catt - Feld]

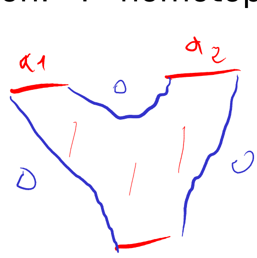
$D \rightsquigarrow$ 

$T\overset{[0,1]}{I} \longrightarrow T^*_\pi M$ cot. paths

$T(I \times I) \longrightarrow T^*_\pi M$ cot. homot.

$\rightsquigarrow \left(\frac{\text{cot paths}}{\text{cot hom}} \right) \rightrightarrows (M, \pi)$

Composition: "Y" homotopies



$a_3 \sim [a_1] \cdot [a_2]$

$TY \longrightarrow T^*_\pi M$

The semiclassical contribution to PSM

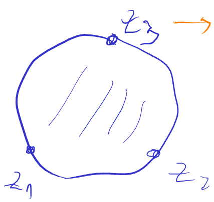
$M \subseteq \mathbb{R}^n$ open "coordinate space"

$$f_1 \star_\hbar f_2(x) = \text{``}\int_{(X,\eta)\in Z}\text{''}\ \underbrace{f_1(X(z_1))}_{e^{\frac{i}{\hbar}p_1 X(z_1)}}\ \underbrace{f_2(X(z_2))}\ \underbrace{\delta_x^M(X(z_3))}_{\int e^{-\frac{i}{\hbar}p_3 X(z_3)}}\ e^{\frac{i}{\hbar}A(X,\eta)}\ \text{``}\mu_\hbar(X,\eta)\text{''}.$$

A'

$$e^{\frac{i}{\hbar}\left(A + p_1 X(z_1) + p_2 X(z_2) - p_3 X(z_3)\right)}$$

D disk



$X : D \to M$

$\eta \in \Omega^1(D, X^! T^* M)$

"fields"

$$A(X,\eta) := \int_D \left[\eta_j \wedge dX^j + \frac{1}{2}\pi^{jk}(X)\eta_j \wedge \eta_k\right]$$

PSM action

[Ik, So-St]

↳ crit points

$(X,\eta) : TD \to T^*_\pi M$

algb. morph.

Heuristically

$$\left(e^{\frac{i}{\hbar}p_1} \star_\hbar e^{\frac{i}{\hbar}p_2}\right)(x) \rightsquigarrow S_p = A'\big|_{\text{crit. pts of } A'}$$

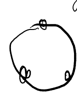
(M,π) coord. $\rightsquigarrow A'$, $(PDE)^\pi$

crit pts of A'

Our main result

(M,π) coordinate Poisson.

- families of solution of $(PDE)^\pi$ exist and their germs are classified by triangles in a canonical local integration $G_\pi \rightrightarrows (M,\pi)$



- for each family of solutions (X,η,p_3),

$$S_p := A'(X,\eta,p_3) \text{ defines a generating function for } G_\pi$$

$\hookrightarrow (p_1,p_2,x)$

integr. (M,π)

$$gr(m) \subseteq \bar{G} \times \bar{G} \times G$$
$$\| $$
$$gr(dS) \qquad G \subseteq T^*M$$

<u>idea</u> describe loc. sympl. gd str. integr. (M,π)

through maps (X,η) on D and A'

The functional A' and the system of PDEs

(M, π) coord Poisson

$$A'(X,\eta,p_3) = \int_D \left[\eta_j \wedge dX^j + \frac{1}{2}\pi^{jk}(X)\eta_j \wedge \eta_k\right] + p_{1j}\delta_{z_1}(X^j) + p_{2j}\delta_{z_2}(X^j) - p_{3j}(\delta_{z_3}(X^j) - x).$$

Dirac's delta dist.

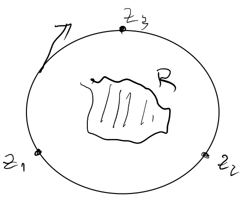
crit pts

$\mathrm{PDE})^{\pi}_{p_1,p_2,x}$

↳ parameters

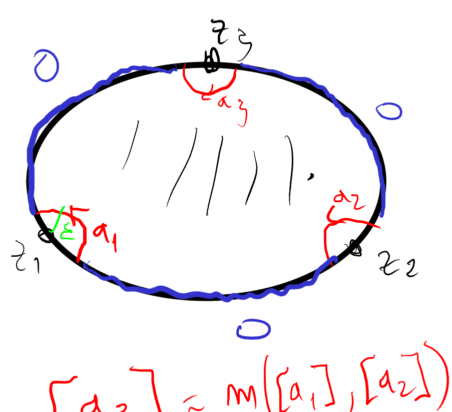
$$\begin{aligned}
d\eta_j + \frac{1}{2}\partial_{x^j}\pi^{ab}(X)\eta_a \wedge \eta_b &= -p_{1j}\delta_{z_1} - p_{2j}\delta_{z_2} + p_{3j}\delta_{z_3} \\
dX^j &= \pi^{ij}(X)\eta_i \text{ at points in } int(D) \\
\delta_{z_3}(X) &= x \\
i^*_{\partial D}\eta &= 0
\end{aligned}$$

BC's



$TR \to T^*_\pi M$ algd

Rmk: singular cotangent "Y" homotopies →



$[a_3] = m([a_1],[a_2])$

Rmk:

$$\boxed{\delta_{z_k}(X) = \lim_{\epsilon\to 0}\frac{1}{\pi}\int_{a(\epsilon)}^{b(\epsilon)} X(z_k + \epsilon e^{i\theta})\, d\theta,}$$

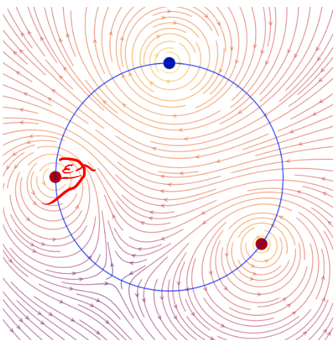
Explicit solutions for π constant

$\pi^{ij}(x) = \pi^{ij}$

eqs for η :
$$\left.\begin{aligned} d\eta &= -p_1\delta_{z_1} - p_2\delta_{z_2} + p_3\delta_{z_3} \\ i^*_{\partial D}\eta &= 0 \\ \wedge(gf) \quad d*\eta &= 0 \end{aligned}\right\} (S)$$

$\int_D$
$\Rightarrow p_3 = p_1 + p_2$

$M = \mathbb{R}$
$\eta \rightsquigarrow$ vector field
D



K propagator

$$\Gamma_{z_0} := \frac{1}{4\pi i}\left[d_z ln\left(\frac{(z-z_0)(1-z\bar{z}_0)}{(\bar{z}-\bar{z}_0)(1-\bar{z}z_0)}\right) + zd\bar{z} - \bar{z}dz\right]$$

$z_0 \in \partial D$, then $i^*_{\partial D}\Gamma_{z_0} = 0$, $\underbrace{d\Gamma_{z_0} = \delta_{z_0} - \frac{1}{\pi}dx \wedge dy}$ and $d \star \Gamma_{z_0} = 0$.

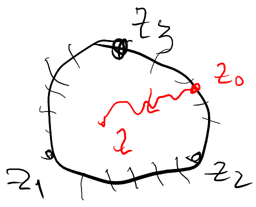
sol (S) $\Rightarrow$ $\eta = -p_1\Gamma_{z_1} - p_2\Gamma_{z_2} + (p_1+p_2)\Gamma_{z_3}$

$$\boxed{\gamma^*_\epsilon\Gamma_{z_0} = \frac{1}{\pi}d\theta + O(\epsilon).(*)}$$

$\rightarrow$ S's on PDE
$\Rightarrow$ beh. of η
near z_i's

Solving for X :

$$X(z) = X(z_0) + \int_\gamma \pi^\sharp\eta,$$



Jumps
$$X^j(\gamma_k(1)) - X^j(\gamma_k(0)) = -\pi^{ij}p_{ki},$$ $k = 1,2,3$

$X|_{\partial D \setminus \{z_1,z_2,z_3\}}$ loc. const.

Rmk: $\Rightarrow A'|_{sol} = (p_1+p_2)x + \frac{1}{2}\pi(p_1,p_2)$ $\quad (\forall (X,\eta)$ sol sat $(*))$

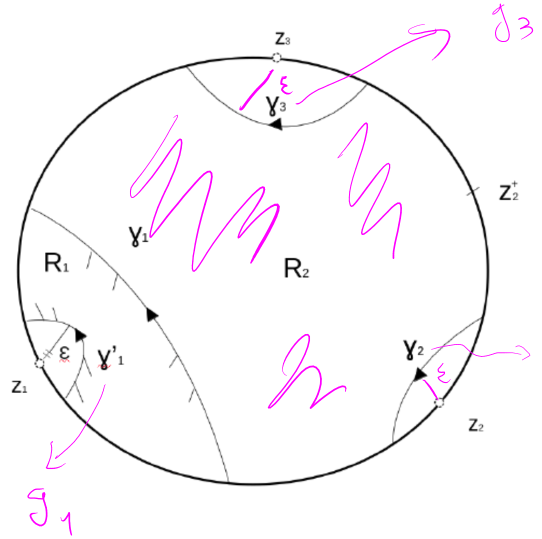
$$X(z_3^+) = x - \frac{1}{2}\pi^\sharp(p_1+p_2) \qquad X(z_3^-) = x + \frac{1}{2}\pi^\sharp(p_1+p_2)$$

$$X(z_1^+) = X(z_2^-) = x + \frac{1}{2}\pi^\sharp(p_1) - \frac{1}{2}\pi^\sharp(p_2)$$

Rmk: $M = \mathbb{R}^2$, $\pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ canonical

$\Longrightarrow$ choose (p_1, p_2, x) $\quad$ $c : D \setminus \{z_1, z_2, z_3\} \to \Delta_2 = \{(t, s) \in \mathbb{R}^2 : t + s \leq 1, t, s \geq 0\},$ $\quad$ parametrization of standard 2-simplex

General solutions: singular homotopies and groupoid elements



singular "Y" homot.

$$g_3 = m(g_1, g_2)$$

p_3 p_1 p_2

g_k's determined by p_k & z_k's on PDE

The canonical (local) integration

(M,π) coord. $\rightsquigarrow$ canon. loc.-sympl. gd G_π [Kar]

$p \sim 0$

$$\int_0^1 du\, \varphi^u_{\pi,p}(\alpha_\pi(x,p)) = x$$

$\omega_c \longrightarrow \xi, \xi_\pi$

$\alpha_\pi : U_\alpha \subset T^*M \to M$

source

$$\alpha_\pi(x,p) = x + O(p)$$

"strict sympl. realization"

[Cos, Daz, We] loc. gd structure

$\beta \longrightarrow$ right invar

$$\phi^{\alpha^* f}_u(z) = m(z, \phi^{\alpha^* f}_u(\alpha(z))),$$

ham flows on (T^*M, ω_c)

$\Longrightarrow$ $\mathrm{inv}(x,p) = (x,-p)$

Technical details about Disks $\leftrightarrow$ Triangles

Defs: "strong" solutions and families

$$z^*_{k,\epsilon}\eta = \sigma_k p_k \frac{d\theta}{\pi} + O(\epsilon)$$ Strong solution on punctured disk

$$U \times TD_* \to T^*M, (p_1, p_2, x, z, \dot{z}) \mapsto (X_{p_1,p_2,x}(z), i_{\dot{z}}\eta^{p_1,p_2,x}|_z, p_3(p_1, p_2, x))$$ family of solutions

$$g : D_* \to \overset{G_\pi}{P_G},\ -\omega^\flat_G(DR_{g^{-1}}dg) = (X, \eta),\ g(z^+_2) = 1_{X(z^+_2)}.$$ integrate punctured disk

$D_* = D \setminus \{z_1, z_2, z_3\}$

Lemma: how the singularities in the algebroid map determine the groupoid disk

$$lim_{\epsilon\to 0}\, g(z_k + \epsilon e^{i\theta}) = \phi^{-\sigma_k p_k \beta_\pi}_{u_k(\theta)}(g(z^+_k)),\ k = 1, 2, 3$$

$$g_3 = m_\pi(g_1, g_2)$$

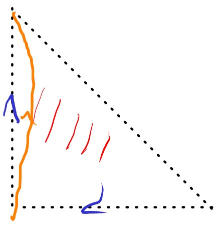
$$g_k = (\delta_{z_k}(X), p_k), k = 1, 2, 3,$$

Def: G_π-triangle

$$\hat{g} : \Delta_2 \to T^*M$$

generated by (p_1, p_2, x)

edges = Right invar flow gen by p_k's $\in \Gamma T^*_\pi M$



(X, η) sol $\xrightarrow{\text{integr.}}$ disk in G_π / : $Lie(G_\pi) \simeq (M, \pi)$

$\downarrow \wr$ change of parameterization $c : D_* \hookrightarrow \Delta_2$

$(*)$ Triangles in G_π

Prop: $\hat{g} \longmapsto -\omega^\flat_c\left(d(\hat{g}\circ c)\,(\hat{g}\circ c)^{-1}\right) \equiv (X, \eta)$ defs 1:1 corresp. of germs

$[G_\pi\text{-triangles}] \longrightarrow [\text{solutions } (PDE)^\pi]$

The generating function property

$G \subseteq_{op} T^*M$, $1_x = 0_x$, $\omega = \omega_c$

$\rightrightarrows M$

loc. sympl. gd ($M \simeq \mathbb{R}^n$)

$$gr(m_G) =_{M^{(3)}} \{((\overbrace{\partial_{p_1} S}^{x_1}, p_1), (\overbrace{\partial_{p_2} S}^{x_2}, p_2), (x, \overbrace{\partial_x S}^{p_3})) : (p_1, p_2, x) \in X\} \subset \overline{T^*M} \times \overline{T^*M} \times T^*M.$$

$S : T^*M \times_M T^*M \to \mathbb{R}$
(p_1, p_2, x)

$T^*M \simeq M \times M^*$

groupoid axioms

The SGA equation [Catt, Dhe, Fel]

$$S(p_1, p_2, \bar{x}) + S(\bar{p}, p_3, x) - \bar{p}(\bar{x}) = S(p_1, \tilde{p}, x) + S(p_2, p_3, \tilde{x}) - \tilde{p}(\tilde{x}),$$

$$\bar{x} = \partial_{p_1} S(\bar{p}, p_3, x), \quad \bar{p} = \partial_x S(p_1, p_2, \bar{x}), \quad \tilde{x} = \partial_{p_2} S(p_1, \tilde{p}, x), \quad \tilde{p} = \partial_x S(p_2, p_3, \tilde{x}).$$

Thm: (AC) G_π (Kar integr) admits a canonical gen. funct. S_π germ uniquely charact. by $S_\pi(0,0,x) = 0$

Back to the main result:

$\Big($ (p_1, p_2, x) param $(\mathrm{PDE})^{\pi}$ close to $0 \times 0 \times M \subset M^* \times M^* \times M$ $\longmapsto$ (X, η, p_3) family of (strong) soluts $\Big)$ $\longmapsto$ $S_P(p_1, p_2, x) := A'(\underbrace{X, \eta, X}_{\text{corresp to } p_1, p_2, x})$

$\hookrightarrow$ is a gen funct for G_π

Ideas:

$$p_3 =_{0 \times 0 \times M} \partial_x S_\pi.$$

$$\begin{aligned} \partial_{p_1} S_P(p_1, p_2, x) &= \delta_{z_1}(X_{p_1, p_2, x}) \\ \partial_{p_2} S_P(p_1, p_2, x) &= \delta_{z_2}(X_{p_1, p_2, x}) \\ \partial_x S_P(p_1, p_2, x) &= p_3(p_1, p_2, x). \end{aligned}$$

Conclussions and outlook

the case of linear Poisson: BCH and gauge theory

The SGA eq. from 3-simplices

general manifolds M: focus on triangles

Formal expansions

Compatible 1/2-densities

other boundary conditions:Morita modules

Other surfaces

Integrability of M and quantization