Symplectic : Contact

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Poisson : Jacobi

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Affine : Projective

based on joint work with M.A. Salazar (arXiv:1406.2138)
& on ongoing joint work with C. Angulo & M.A. Salazar.

The Main Characters (1)

Symplectic (S,ω)

$\omega \in \Omega^2(M)$ s.t. $d\omega = 0$ &

$\omega^\flat : TM \xrightarrow{\cong} T^*M$

Contact (C,H)

$H \subset TC$ codim 1 s.t.

$[\cdot,\cdot]|_H$ mod H non-deg.

$L = TC/H \to C$

$H \times H \to L$

The Main Characters (2)

Poisson $(P, \{\cdot,\cdot\})$

$\{\cdot,\cdot\}$ Lie bracket on $C^\infty(P)$

s.t.

$\forall f \in C^\infty(P)$ $\{f, -\} \in \mathfrak{X}(P)$

$(\mathfrak{g}^*, \{\cdot,\cdot\}_{lin})$

Jacobi $(L \to M, \{\cdot,\cdot\})$

$L \to M$ line bundle &

$\{\cdot,\cdot\}$ local Lie bracket on $\Gamma(L)$

$\forall u, v \in \Gamma(L)$

$\operatorname{supp}\{u,v\} \subset \operatorname{supp} u \cap \operatorname{supp} v.$

$(\mathcal{O}(1) \to \mathbb{P}(\mathfrak{g}^*), \{\cdot,\cdot\})$

Interlude: (G,X) structures

$G \curvearrowright X$ **rigid** if $U \subset X$ open & $g|_U = \mathrm{id}|_U \Rightarrow g = e$.

(G,X) structure on B: atlas $\mathcal{A} = \{(U_\alpha, \phi_\alpha : U_\alpha \to \phi_\alpha(U_\alpha) \subset X)\}$ s.t.
$\forall \alpha, \beta$ with $U_\alpha \cap U_\beta \neq \emptyset$ $\exists g \in G$ s.t. $\phi_\beta \circ \phi_\alpha^{-1} = g|_{\phi_\alpha(U_\alpha \cap U_\beta)}$.

<u>Equivalently</u>:

(i) local diffeo dev: $\tilde{B} \to X$,

(ii) $\rho : \pi_1 B \longrightarrow G$ s.t. dev $\pi_1 B$-equivariant.

The Main Characters (3)

$\mathbb{Z}$-Affine $(B,\mathcal{A})$

$\mathcal{A} = (\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n), \mathbb{R}^n)$ str.

$\mathrm{Aff}(\mathbb{R}^n) = GL(n;\mathbb{R}^n) \ltimes \mathbb{R}^n$

$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) = GL(n;\mathbb{Z}) \ltimes \mathbb{R}^n$

$\mathbb{Z}$-Projective $(B,\mathcal{A})$

$\mathcal{A} = (PGL^+_{\mathbb{Z}}(\mathbb{R}^{n+1}), \mathbb{R}P^n)$ str.

$PGL^+(\mathbb{R}^{n+1}) = GL(n+1;\mathbb{R})/\mathbb{R}_{>0}$

$PGL^+_{\mathbb{Z}}(\mathbb{R}^{n+1}) = GL(n+1;\mathbb{Z})$

$\mathrm{dev}^*\mathcal{O}(1) \longrightarrow \tilde{B}$

$\mathrm{dev}: \tilde{B} \longrightarrow \mathbb{R}P^n$

<u>Why?</u>

- Arnol'd (1989): Symplectic $\rightleftarrows$ Affine, Contact $\rightleftarrows$ Projective

$$\begin{array}{cc} \text{Symplectic} & \text{Contact} \\ \updownarrow & \updownarrow \\ \text{Affine} & \text{Projective} \end{array}$$

- Recent interest in Contact/Jacobi
- "Compactness easier in Contact/Jacobi/Projective"

→ Borman, Eliashberg, Murphy

→ Contrast: (Milnor) Σ^2 closed affine $\Leftrightarrow \chi(\Sigma) = 0$.

(Benzécri) $\forall\, \Sigma^2$ closed (Σ, A) proj.

Homogenisation

T^*C
$\cup$

$$(L^* \setminus 0, \omega_{can}|_{L^*\setminus 0}) \longleftarrow (C, H) \quad L = TC/H$$

$$(L^* \setminus 0, \{\cdot,\cdot\}) \longleftarrow (L \to M, \{\cdot,\cdot\})$$

$$(L^* \setminus 0, \hat{A}) \longleftarrow (B, A)$$

$$\rho : \pi_1 B \to PGL^+(\mathbb{R}^{n+1})$$

$$\uparrow \qquad\qquad \downarrow$$

$$\pi_1(L^* \setminus 0) \qquad Aff(\mathbb{R}^{n+1})$$

$$\tilde{L} := dev^* O(1)$$

$$\tilde{L}^* \setminus 0 \cong dev^*(O(-1) \setminus 0)$$

$$dev : \tilde{L}^* \setminus 0 \to O(-1) \setminus 0 \to \mathbb{R}^{n+1} \setminus 0$$

Inclusions

$$(P, \{\cdot,\cdot\}) \longrightarrow (P \times \mathbb{R} \to P, \{\cdot,\cdot\})$$

$$(B, \mathcal{A}) \longrightarrow (B, \mathcal{A}')$$

$$\mathrm{Aff}(\mathbb{R}^n) \longrightarrow GL(n+1; \mathbb{R})$$

$$(A, \underline{b}) \longmapsto \left(\begin{array}{c:c} A & \underline{b} \\ \hline 0 & 1 \end{array}\right)$$

$$\mathrm{Aff}_{\mathbb{Z}}(\mathbb{R}^n) \not\longrightarrow GL(n+1; \mathbb{Z})$$

$$\mathbb{R}^n \hookrightarrow \mathbb{R}P^n$$

$$x \longmapsto [x : 1].$$

Pre-quantisation (?)

(S,ω), $[\omega] \in H^2(S;\mathbb{Z})$ $\longrightarrow$ $(C, H = \operatorname{Ker}\theta)$

$S^1 \to C \to S$ $\quad [\omega]$, $\theta \in \Omega^1(C)$ connection 1-form

(B, A) strong $\mathbb{Z}$-affine $\longrightarrow$ (B, A') $\mathbb{Z}$-projective

$(\mathrm{Aff}(\mathbb{Z}^n), \mathbb{R}^n)$

$\mathrm{Aff}(\mathbb{Z}^n) = GL(n;\mathbb{Z}) \ltimes \mathbb{Z}^n$

A simple but nice observation

Theorem (Ovsienko) B simply connected, $\dim B = K$

- $K = 2n$: B affine $\Leftrightarrow \exists f_1, \dots, f_{2n} \in C^\infty(B)$ s.t. $\sum_{j=1}^{n} df_{2j-1} \wedge df_{2j}$ symp.

  $\mathrm{dev} : B \to \mathbb{R}^{2n}$

- $K = 2n+1$: B proj $\Leftrightarrow \exists f_1, \dots, f_{2n+2} \in C^\infty(B)$ s.t. $\mathrm{Ker}\left(\sum_{j=1}^{n+1} f_{2j-1} df_{2j} - f_{2j} df_{2j-1} \right)$ contact.

  $\mathrm{dev} : B \to \mathbb{RP}^n$

- B proj $\Leftrightarrow \exists f_1, \dots, f_{K+1} \in C^\infty(B)$ s.t. $\sum_{j=1}^{K+1} (-1)^j f_j \, df_1 \wedge \dots \wedge \widehat{df_j} \wedge \dots \wedge df_{n+1}$ volume

Lagrangian foliations

Lagrangian foliation $\mathcal{F}$ on (S, ω) : $\forall x \in S$ $T_x\mathcal{F}$ Lagrangian.

Theorem (Weinstein)

B leaf of Lagrangian foliation $\iff$ B affine.

Proof.

($\Rightarrow$) : local model $T^*V \to V$ V vector space.

($\Leftarrow$) : Lag foliation $\mathcal{F}_0$ on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times (\mathbb{R}^n)^*$

$\to$ Lag fol $\tilde{\mathcal{F}}$ on $T^*\tilde{B} \cong \mathrm{dev}^* T^*\mathbb{R}^n$ $\mathbb{R}^n \times U$.

Interlude: Legendrians, jets & projectivisation (1)

$Y \xhookrightarrow{i} (C,H)$ Legendrian: $Di(TY) \subset H$ & $Di(TY)$ Lag.

Local model: $L \to Y$ line bundle, $J^1L \to Y$ first jet bundle,

$Y \xhookrightarrow{0} (J^1L, H_{can})$ Cartan contact structure

$$J^1_x L = \{ j^1_x u \; : u \in \Gamma(L) \}.$$

$L = Y \times \mathbb{R} \to Y$ $\qquad H_{can} = \ker(dt - \lambda_{can})$

Interlude: Legendrians, jets & projectivisation (2)

Given $Y \rightsquigarrow \mathbb{P}(T^*Y) \to Y$ projectivisation of $T^*Y \to Y$

$(\mathbb{P}(T^*Y), H_{can} = \text{"}\operatorname{Ker} \lambda_{can}\text{"})$ contact &

$\forall x \in Y \quad \mathbb{P}(T_x^*Y) \hookrightarrow (\mathbb{P}(T^*Y), H_{can})$ Legend.

Legendrian foliations

Legendrian foliation $\mathcal{F}$ on (C, H) : $\forall x \in C$ $T_x\mathcal{F}$ Legendrian

Theorem (Pang, S. ?)

B leaf of Legendrian foliation $\Leftrightarrow$ B projective

Proof.

$(\Rightarrow)$ local model is $\mathbb{P}(T^*V) \to V$

$(\Leftarrow)$ same as before : $J^1\mathcal{O}(1) \cong \mathbb{RP}^n \times \mathbb{R}^{n+1}$

$$\begin{array}{c} \downarrow \\ \mathbb{RP}^n \end{array} \qquad \{\mathbb{RP}^n \times \mathrm{pt}.\}$$

Isotropic realisations of Poisson manifolds (1)

$$\begin{array}{c} (S,\omega) \\ \downarrow \phi \\ (P,\{\cdot,\cdot\}) \\ \downarrow \\ B = P/\mathcal{F} \end{array}$$

isotropic realisation : ϕ Poisson, surj subm & iso fibres

$\implies (P,\{\cdot,\cdot\})$ regular

Isotropic realisations of Poisson manifolds (2)

$$\begin{array}{c} (S,\omega) \\ \downarrow \phi \\ (P,\{\cdot,\cdot\}) \\ \vdots \\ \downarrow \\ B = P/\mathcal{F} \end{array}$$

isotropic realisation : ϕ Poisson, surj subm & iso fibres.

$\Longrightarrow$ $(P,\{\cdot,\cdot\})$ regular

Theorem (Duistermaat, Dazord & Delzant, ...)

If fibres of ϕ compact & connected

(a) "$B = P/\mathcal{F}$ $\mathbb{Z}$-affine"

(b) $\mathbb{Z}$-affine structure + coho class $\rightsquigarrow$ classification

Isotropic realisations of Jacobi manifolds (1)

$$\begin{array}{c} (C, H) \\ \downarrow \phi \\ (L \to M, \{\cdot,\cdot\}) \\ \vdots \\ \downarrow \\ B = M/\mathcal{F} \end{array}$$

isotropic realisation: ϕ Jacobi, surj subm., $\forall x \in M$ $\forall y \in \phi^{-1}(x)$ $T_y \phi^{-1}(x) + H_y = T_y C$ & $T_y \phi^{-1}(x) \cap H_y$ iso.

$G \curvearrowright (C, H)$ w. H fn $\exists\, \phi : C \longrightarrow P(\mathfrak{g}^*)$ s.t.

Salazar & S.

Isotropic realisations of Jacobi manifolds (2)

isotropic realisation : ϕ Jacobi, surj subm., ...

Theorem (Salazar & S.)

(1) $(L \to M, \{\cdot,\cdot\})$ regular & leaves even dim^{l}

If fibres of ϕ compact & connected

$$\begin{tikzcd} (S,\omega) \arrow[dr] & & (C,\theta) \arrow[ll] \arrow[dl] \\ & (P,\pi) & \end{tikzcd}$$

(2) "$B = M/\mathcal{F}$ $\mathbb{Z}$-projective"

(3) $\mathbb{Z}$-projective str. + coho class $\rightsquigarrow$ classification

(4) Pre-quantisation "commutes" with iso realisations.

PMCTs (1)

$$\begin{array}{c}(S,\omega)\\ \downarrow\downarrow\\ (P,\{\cdot,\cdot\})\\ \downarrow\\ B=P/\mathcal{F}\end{array}$$

PMCT: $\exists$ "compact" symplectic integration.

Examples: compact symplectic, $\mathbb{Z}$-affine, duals of compact Lie algebras, ...

PMCTs (2)

PMCT: $\exists$ "compact" symplectic integration.

<u>Theorem</u> (Crainic, Fernandes, Martínez-Torres)

$(P, \{\cdot,\cdot\})$ regular PMCT.

(a) $B = P/\mathcal{F}$ $\mathbb{Z}$-affine orbifold

(b) $\mathbb{Z}$-affine $\leftrightarrows$ Poisson structure

(c) symplectic gerbe $\cdots\cdots\rightarrow$ existence iso realisations of $(P, \{\cdot,\cdot\})$.

<u>Why JMCTs?</u>

<u>Theorem</u> (Marcut)

$\mathfrak{g}$ compact, semisimple. Complete local description of

$$\mathrm{Poisson}\left(\mathcal{S}(\mathfrak{g}^*), \{\cdot,\cdot\}_{\mathcal{S}(\mathfrak{g}^*)}\right).$$

<u>Obs.</u> 1) $\left(\mathcal{S}(\mathfrak{g}^*), \{\cdot,\cdot\}_{\mathcal{S}(\mathfrak{g}^*)}\right)$ <u>not</u> integrable in general

2) But $\left(\mathcal{S}(T^*G), \pi_{can}\right) \Rightarrow \left(\mathcal{S}(\mathfrak{g}^*), \{\cdot,\cdot\}_{\mathcal{S}(\mathfrak{g}^*)}\right)$

compact integration.

JMCTs (1)

$$\begin{tikzcd} (C,H) \arrow[d, shift left] \arrow[d, shift right] \\ (L \to M, \{\cdot,\cdot\}) \arrow[d] \\ B = M/\mathcal{F} \end{tikzcd}$$

JMCT : $\exists$ "compact" contact integration

<u>Examples</u> : $\mathbb{Z}$ compact symplectic manifolds, $\mathbb{Z}$-projective manifolds, projectivisation of duals of compact Lie algebras, PMCT with $\mathbb{Z}$ symplectic form, ...

JMCTs (2)

JMCT : $\exists$ "compact" contact integration

<u>Theorem</u> (Angulo & Salazar & S.)

(1) $(L \to M, \{\cdot,\cdot\})$ JMCT $\Rightarrow$ $(L \to M, \{\cdot,\cdot\})$ "$\cong$" Poisson

(2) $(L \to M, \{\cdot,\cdot\})$ regular JMCT $\Rightarrow$ $B = M/\mathcal{F}$ $\mathbb{Z}$-projective orbifold.

Next steps & Questions

- Pre-quantisation "commutes" with being of compact type?
- Rigidity results?
- Construction of examples of JMCTs?
- Geometry of Legendrian foliations?