Vortex sheets in ideal fluids and coadjoint orbits

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- inspired by works of J. Goldin, R. Menikoff, D. Sharp and B. Khesin
- special thanks to S. Haller (lemma)

Coadjoint orbits of $\mathrm{Diff}_{vol}(\mathbb{R}^3)$

$\mu = dv$ euclidean volume form

$G = \mathrm{Diff}_{vol}(\mathbb{R}^3) = \{ \varphi \in \mathrm{Diff}_c(\mathbb{R}^3) : \varphi^* \mu = \mu \}$

$\mathfrak{g} = \mathfrak{X}_{vol}(\mathbb{R}^3) = \{ u \in \mathfrak{X}_c(\mathbb{R}^3) : \operatorname{div} u = 0 \}$

Regular dual: $\mathfrak{g}^*_{reg} = \Omega^1(\mathbb{R}^3) / \ker d \simeq d\Omega^1(\mathbb{R}^3)$

- each $u \in \mathfrak{g}$ admits a potential 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$

$$u = X_\alpha \iff i_u \mu = d\alpha$$

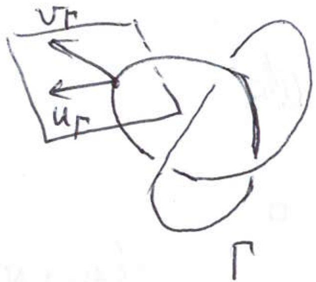
- $X_\alpha \longmapsto \int_{\mathbb{R}^3} \alpha \wedge d\sigma$ is $d\sigma \in \mathfrak{g}^*_{reg}$

Knot spaces as coadjoint orbits

$\Gamma \in Gr^{S^1}(\mathbb{R}^3)$ Fréchet mfd. of closed oriented curves

Principal $\mathrm{Diff}_+(S^1)$ - bundle

$$\mathrm{Emb}(S^1, \mathbb{R}^3) \longrightarrow Gr^{S^1}(\mathbb{R}^3)$$
$$f \longmapsto f(S^1)$$



Marsden - Weinstein sympl. form

$$\Omega_\Gamma(u_\Gamma, v_\Gamma) = \int_\Gamma i_{v_\Gamma} i_{u_\Gamma} \mu \quad , \quad u_\Gamma, v_\Gamma \in \Gamma(T\Gamma^\perp)$$

Singular moments in $\mathfrak{g}^*$

$$\begin{array}{ccc} \Gamma & \longmapsto & \left(X_\alpha \longmapsto \int_\Gamma \alpha \right) \\ \cap & & \cap \\ (Gr^{S^1}(\mathbb{R}^3), \Omega) & \hookrightarrow & \mathfrak{g}^* \end{array}$$

Thm: Its connected components are coadjoint orbits of G_0.

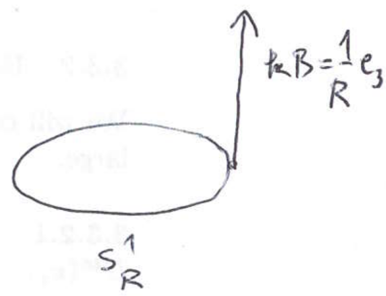
[Haller-V.] More general codim. 2 + Lichnerowicz cocycles.

Vortex filament equation (VFE)

Hamilton equation on coadj.-orb. $G_2^{S^1}(\mathbb{R}^3)$

$h(\Gamma) = \text{Length}(\Gamma)$

$X_h(\Gamma) = kB$ binormal curvature



(Da Rios (L.I.A.), Hasimoto (NLS), Langer-Perline, Calini) completely integrable

(VFE) $\boxed{\frac{d}{dt}\Gamma = kB}$

In higher dim. (but codim. 2) h is the volume functional

$X_h = -\text{Rot}^{\perp}_{90^\circ}(\underbrace{\text{Trace}\, II}_{H})$ skew mean curvature

(Haller-V., Khesin, Shashikanth, Song + Yang (Hasimoto transf).

Surfaces as coadjoint orbits

Model surface: S closed oriented

$$\mathrm{Emb}(S, \mathbb{R}^3) \to \mathrm{Gr}^S(\mathbb{R}^3) \quad \text{principal } \mathrm{Diff}_+(S)\text{-bundle}$$

$$f \longmapsto f(S)$$

$\mathrm{Gr}^S(\mathbb{R}^3)$ Fréchet mfd. of oriented surfaces $\Sigma \subset \mathbb{R}^3$ of type S

$$T_\Sigma \mathrm{Gr}^S(\mathbb{R}^3) = \Gamma(T\Sigma^\perp) \equiv C^\infty(\Sigma)$$

G - action:

$$\varphi \cdot \Sigma = \varphi(\Sigma)$$

Infinitesimal generator

$$\zeta_u(\Sigma) = u\big|_\Sigma^\perp \equiv n \cdot u$$

preserves the enclosed volume $a = \int_\Sigma \nu$ where $\mu = d\nu$.

$$T_\Sigma \mathrm{Gr}_a^S(\mathbb{R}^3) = C_0^\infty(\Sigma) \ni n \cdot u$$

by the divergence thm flux of u through Σ is zero.

Vortex sheets / vortex lines

Need 1-forms attached to the surface (Khesin '12)

$$(\Sigma, \beta_\Sigma) \longmapsto \left(X_\alpha \longmapsto \int_\Sigma \alpha \wedge \beta_\Sigma\right)$$

$$\begin{array}{ccc} \cap & & \cap \\ Gr^{S,\beta}(\mathbb{R}^3) & \hookrightarrow & \mathfrak{g}^* \end{array}$$

$\Rightarrow$ decorated Grassmannians (e.g. weighted Lag. submfds.)
Ham

Vorticity density $\beta_\Sigma \in \Omega^1(\Sigma)$

closed, with singularities (finite nr. of zeros) of Morse type

Vortex lines = leaves of the singular foliation

$\ker \beta_\Sigma$

Ask for singularities of Morse type
(locally the log. deriv. of a Morse function)

Vector field approach

Vorticity density $\gamma_\Sigma \in \mathfrak{X}(\Sigma)$ (Goldin '91, Menikoff & Sharp)

$\operatorname{div} \gamma_\Sigma = 0$ for μ_Σ

Relation to β_Σ:

$$\gamma_\Sigma = \mathrm{Rot}^{\Sigma}_{90^\circ}(\beta_\Sigma^{\#}) \text{ i.e. sgrad of } \beta_\Sigma$$

Thus

$$(\Sigma, \gamma_\Sigma) \longmapsto \left(u \mapsto \int_\Sigma (\gamma_\Sigma \cdot X)\, \mu_\Sigma \right) \in \mathfrak{g}^*$$

where X is a vector potential of $u \in \mathfrak{g}$ $(= \alpha^{\#})$

Advantages of the 1-form approach:

- no ambient metric involved
- use pushforward of $\beta \in \Omega^1(S)$ closed (no given volume form on S)

Tangent 'space' at (Σ, β_Σ)

Forgetful map

$$\mathrm{Gr}^{S,\beta}(\mathbb{R}^3) \longrightarrow \mathrm{Gr}^S(\mathbb{R}^3)$$
$$(\Sigma, \beta_\Sigma) \longmapsto \Sigma$$

is an associated bundle to the principal bundle

$$\mathrm{Emb}(S, \mathbb{R}^3) \to \mathrm{Gr}^S(\mathbb{R}^3)$$

for the $\mathrm{Diff}_+(S)$-action on 1-forms restricted to

$$\mathcal{O}_\beta = \mathrm{Diff}_+(S) \cdot \beta \qquad (\text{with tg. space } d\,C^\infty_\beta(S))$$

Principal and associated
Connections induced by the euclidean metric

$\rightsquigarrow$ hor/ver splitting

$$T_{(\Sigma,\beta_\Sigma)} \mathrm{Gr}^{S,\beta}(\mathbb{R}^3) = T_\Sigma \mathrm{Gr}^S(\mathbb{R}^3) \times d\,C^\infty_{\beta_\Sigma}(\Sigma)$$
$$\zeta_u^{(\Sigma,\beta_\Sigma)} = \left(u|_\Sigma^\perp , \; d\beta_\Sigma(u|_\Sigma^T) \right)$$

Symplectic form Ω on vortex sheets

Descends from $\omega = \widehat{\mu \cdot \beta}$ on $\mathrm{Emb}(S, \mathbb{R}^3)$

the principal $\mathrm{Diff}_+(S, \beta)$-bundle

$\mathrm{Emb}_a(S, \mathbb{R}^3) \longrightarrow Gr_a^{S,\beta}(\mathbb{R}^3)$

Here $a = \int_S f^* \nu$, $\quad f \longmapsto (f(S), f_* \beta)$

Non-degenerate, where $d\nu = \mu$.

$(\rho, d\lambda) = \int_\Sigma \rho \lambda \, \mu_\Sigma$ pairing on $C_0^\infty(\Sigma) + dC^\infty_\beta$...

Symplectic form Ω (not exact, even ...

$\Omega_{(\Sigma, \beta_\Sigma)}((\rho_1, d\lambda_1), (\rho_2, d\lambda_2 ...$

Symplectic form Ω on vortex sheets

Descends from $\omega = \widehat{\mu \cdot \beta}$ on $\mathrm{Emb}(S, \mathbb{R}^3)$ via the principal $\mathrm{Diff}_+(S, \beta)$-bundle

$$\mathrm{Emb}_a(S, \mathbb{R}^3) \longrightarrow Gr_a^{S,\beta}(\mathbb{R}^3)$$
$$f \longmapsto (f(S), f_* \beta)$$

Here $a = \int_S f^* \nu$, where $d\nu = \mu$.

Non-degenerate pairing on $C_0^\infty(\Sigma) \times d\, C^\infty_{\beta_\Sigma}(\Sigma)$

$$(\rho, d\lambda) = \int_\Sigma \rho \lambda \, \mu_\Sigma$$

finite nr. of zeros

Symplectic form Ω (not exact, even though ω exact)

$$\Omega_{(\Sigma, \beta_\Sigma)}((\rho_1, d\lambda_1), (\rho_2, d\lambda_2)) = (\rho_1, d\lambda_2) - (\rho_2, d\lambda_1)$$

canonical form

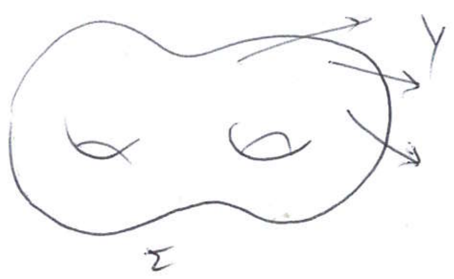
Theorem 1. Connected components of

$$(Gr_a^{S,\beta}(\mathbb{R}^3), \Omega) \hookrightarrow \mathfrak{g}^*$$

are coadjoint orbits of G_0.

Lemma: $Emb_a(S, \mathbb{R}^3)$ is acted on transitively by $\mathfrak{g}$, the Lie alg. of divergence free v.f.s.

Relies on an extension property (learned from Haller):



Any $Y \in \mathfrak{X}(\Sigma)$ can be extended to $X_\alpha \in \mathfrak{g}$ with $i_\Sigma^* \alpha = 0$

Fibration by vortex lines

$\beta = \delta b$ for a fiber bundle projection (i.e. $\beta = b^* dz$)

$$b : S \to \mathbb{T}_\ell = \mathbb{R}/\ell\mathbb{Z}$$

$\Leftrightarrow$ β has no zeroes and $\mathrm{Per}(\beta) = \ell\mathbb{Z}$ discrete subgroup of $\mathbb{R}$.

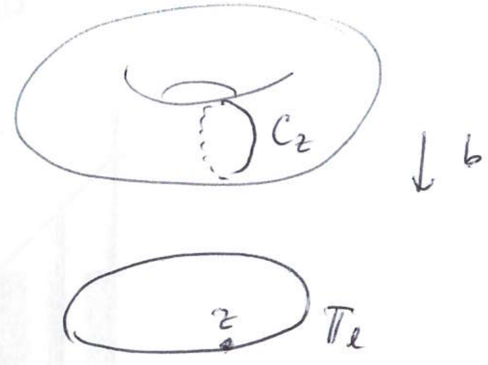
$\Rightarrow$ Only surfaces of genus zero are allowed!

- Vortex lines $C_z = b^{-1}(z)$

- Unique b

up to rotations in $\mathbb{T}_\ell$:

$$\delta(zb) = \delta(b), \quad z \in \mathbb{T}_\ell$$

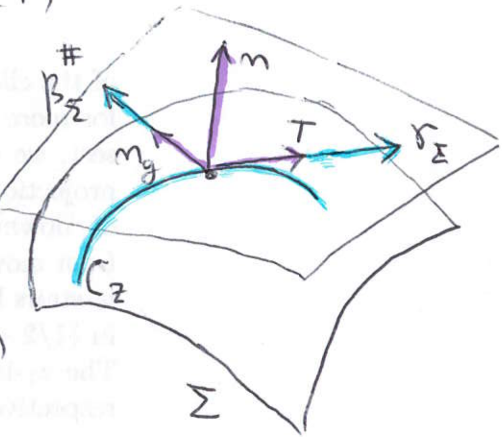


circle of length ℓ

Hamiltonian function

Proposed by (Khesin '12)

$$h(\Sigma, \beta_\Sigma) = \int_{\mathbb{T}_\ell} \underline{\mathrm{Length}(C_z)}\, dz \quad (1)$$



- Darboux frame $\{T, m_g, m\}$

$$\gamma_\Sigma = \underbrace{\beta_\Sigma(m_g)}_{>0}\, T$$

Fubini: $h(\Sigma, \beta_\Sigma) = \int_\Sigma \underline{\beta_\Sigma(m_g)}\, \mu_\Sigma \quad (2)$

V.f. approach: $h(\Sigma, \gamma_\Sigma) = \int_\Sigma \underline{\|\gamma_\Sigma\|}\, \mu_\Sigma \quad (3)$

- Frenet frame $\{T, N, B\}$

$$kN = k_g m_g + k_n m$$

Vortex sheet equation (VSE)

Ham. v.f. on $\mathrm{Gr}_a^{S,\beta}(\mathbb{R}^3)$:

$$X_h(\bar\Sigma, \beta_\Sigma) = \Big(k_g n\,,\; -d\big(k_n \underbrace{\beta_\Sigma(n_g)}_{\|\gamma_\Sigma\|}\big)\Big)$$

in $C_0^\infty(\Sigma) \times d\,C^\infty(\Sigma)$ since β_Σ without zeros

$$\overset{\psi}{k_g} = -\operatorname{div} n_g \quad \text{w.r.t. } \mu_\Sigma .$$

In terms of binormal curvature: $kB = k_g n - k_n n_g$

$$X_h(\Sigma, \beta_\Sigma) = \left(k B^{\perp\Sigma},\; d\,\beta_\Sigma\left(k B^{T\Sigma}\right)\right)$$

(VSE) $\dfrac{d}{dt}(\bar\Sigma, \beta_\Sigma) = \Big(k_g n\,,\; -d\big(k_n \beta_\Sigma(n_g)\big)\Big)$

cannot be splitted in general (on $\mathcal{B}_{1,0}$)

Equations for parametrizations

Essential difference between (VFE) and (VSE)

The euclidean metric on $\mathbb{R}^3$ induces

a principal connection

on $\mathrm{Emb}(S, \mathbb{R}^3) \longrightarrow G^S(\mathbb{R}^3)$ with group $\mathrm{Diff}_+(S)$,

but not on $\mathrm{Emb}(S, \mathbb{R}^3) \longrightarrow G^{S,\beta}(\mathbb{R}^3)$ with group $\mathrm{Diff}_+(S,\beta)$.

Thus $X_h(\Gamma) = kB$ can be horizontally lifted

to a vector field on Emb.

$$\leadsto \quad \frac{d}{dt} f = \frac{f' \times f''}{|f'|^3} \qquad \text{(VFE)},$$

while $X_h(\Sigma, \beta_\Sigma) = \left(kB^{\perp\Sigma}, d\beta_\Sigma(kB^{\top\Sigma})\right)$ cannot!

for (VSE)

(No) Stationary points ?

Starting with the geodesic fibration of the torus of revolution Σ, at $t=0$ by meridians

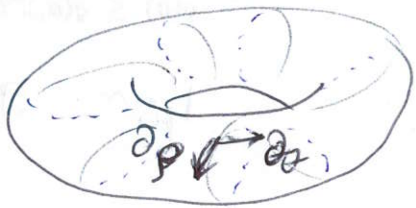
the surface Σ changes shape for $t>0$!

$\rho, \theta \in \mathbb{T}_{2\pi}$

$$\Sigma : \left((R + r\cos\rho)\cos\theta,\ (R + r\cos\rho)\sin\theta,\ r\sin\rho \right)$$

$\beta_\Sigma = d\theta \quad$ period group $2\pi\mathbb{Z}$

$$k_g = 0 \quad \Rightarrow \begin{cases} k_n = k = \dfrac{1}{r} \\ B = n_g \end{cases}$$



Thus

$$X_h(\Sigma, d\theta) = \left(0, \frac{1}{r} d\left(\frac{1}{R + r\cos\rho}\right)\right) \qquad (\Sigma, \underbrace{d\theta}_{\beta_\Sigma}) \in R_{0,1}$$

or for $t>0$ the fibration doesn't stay geodesic

Surfaces of revolution

The rotation invariant subset of $\mathcal{G}_\pi^{S,\beta}(\mathbb{R}^3)$ is

$$\mathcal{R} = \{ (\Sigma, \beta_\Sigma) : \Sigma \text{ surface of revolution}, \ \Phi_\theta^* \beta_\Sigma = \beta_\Sigma \} \quad \forall \theta \in \mathbb{T}_{2\pi}$$

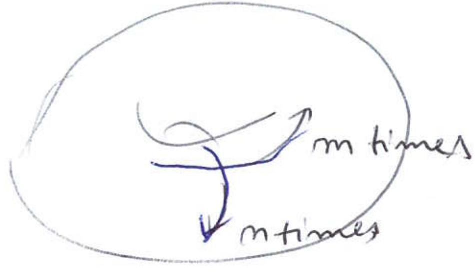
Hence $b_\Sigma \circ \Phi_\theta = z_\theta b_\Sigma$, $z_\theta \in \mathbb{T}_\ell$

so $\Phi_\theta (C_z) = C_{z_\theta z}$: one fiber C_z gives all other fibers by rotation.

Union of connected components

$$\mathcal{R} = \underbrace{\bigcup_{m,n \text{ coprime}} \mathcal{R}_{m,n}}$$

$$\beta_\Sigma = \underbrace{\partial_\rho \zeta \, d\rho}_{m\ell\mathbb{Z}} + \underbrace{c\, d\theta}_{n\ell\mathbb{Z}} \quad \text{period } \ell\mathbb{Z}$$



Σ

for $\zeta(\rho + 2\pi) = \zeta(\rho) + m\ell$

Hamiltonian equation on $\mathcal{R}_{1,0}$

(Fibration by parallel circles)

Both h (Ham fct) and Ω (sympl. form) are invariant under the euclidean group $SE(3)$, thus under axial rotations

$\Rightarrow$ Ham. v.f. X_h is tangent to $\mathcal{R}_{1,0}$.

$\rho = \mathrm{const}$

Seen also via computation:

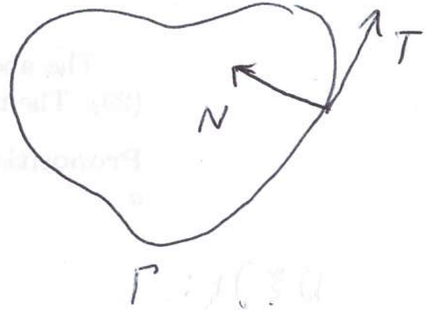
$\Gamma : (\xi(\rho), \eta(\rho))$ and $\beta_\Sigma = \partial_\rho \zeta \, d\rho$

$$X_h(\Sigma, \beta_\Sigma) = \left(- \frac{\partial_\rho \xi}{\xi \sqrt{\partial_\rho \xi^2 + \partial_\rho \eta^2}} , - d\left(\frac{\partial_\rho \eta \, \partial_\rho \zeta}{\xi (\partial_\rho \xi^2 + \partial_\rho \eta^2)} \right) \right)$$

doesn't depend on θ variable $= (k_g n, -d(k_n \beta_\Sigma(m_\sharp)))$

Rem: First component doesn't involve $\partial_\rho \zeta$!

Evolution equation for plane curves



Thm 2 $\dfrac{d}{dt}\Gamma = k_g N \quad (\text{VSE on } \mathcal{B}_{1,0})$

where $N = n|_\Gamma$ unit normal to the plane curve Γ,

and k_g = geodesic curvature of the parallel circle $= -\frac{1}{\xi} D_\rho \xi$

where $D_\rho \xi = \dfrac{\partial_\rho \xi}{\sqrt{\partial_\rho \xi^2 + \partial_\rho \eta^2}}$ arclength derivative

$N = (-D_\rho \eta, D_\rho \xi)$

$$X_h(\Sigma, \beta_\Sigma) = \left(-\frac{D_\rho \xi}{\xi}, -d\left(\frac{D_\rho \eta D_\rho \xi}{\xi} \right) \right)$$ independent of parametrization

Can be lifted to $\mathrm{Emb}(S^1, \mathbb{R}^2) \ni f = (\xi, \eta)$ param. curves

$$\frac{d}{dt} f = (k_g N) \circ f$$

Rem: Doesn't preserve arclength

Constants of motion

(VSE on $\mathcal{B}_{1,0}$) $\begin{cases} \partial_t \xi = \dfrac{D_\rho \xi \, D_\rho \eta}{\xi} \\ \partial_t \eta = -\dfrac{(D_\rho \xi)^2}{\xi} \end{cases}$ plus the vorticity density eq. $\partial_t \zeta = -\dfrac{D_\rho \eta \, D_\rho \zeta}{\xi}$

- Hamiltonian function: $2\pi \int_0^{2\pi} \xi \, \partial_\rho \zeta \, d\rho$
- Enclosed volume: $a = \pi \int_0^{2\pi} \xi^2 \partial_\rho \eta \, d\rho$
- Momentum associated to the translation action in direction of the rotation axis Oz (commutes with axial rotations):

$$\pi \int_0^{2\pi} \xi^2 \partial_\rho \zeta \, d\rho = \int_0^{2\pi} \mathrm{Area}(D_z) \, dz \, , \quad \partial D_z = C_z \, .$$

(euclidian disk)

Prequantization

$\beta = \delta b$

The $\mathrm{Diff}_+(S)$ orbit of b, denoted $\mathcal{O}_b$, depends only on β.

Build again an associated bundle

$$Gr_{ex}^{S,\beta}(\mathbb{R}^3) := \mathrm{Emb}(S,\mathbb{R}^3) \times_{\mathrm{Diff}_+(S)} \mathcal{O}_b \ni (\Sigma, b_\Sigma)$$

and $\mathrm{Emb}(S,\mathbb{R}^3) \to Gr_{ex}^{S,\beta}(\mathbb{R}^3)$ principal $\mathrm{Diff}_+(S,b)$-bundle

$= \mathrm{Diff}_{ex}(S,\beta)$

$= \ker\left(c_\beta : \mathrm{Diff}_+(S,\beta) \to \mathbb{T}_\ell\right)$

$\varphi \mapsto \frac{b \circ \varphi^{-1}}{b}$

surjective (flux homom.)

$\mathfrak{X}_{ex}(S,\beta) = \{v : \beta(v) = 0\}$

$\cap$

$\mathfrak{X}(S,\beta) = \{v ; \beta(v) = \mathrm{const.}\}$

Lemma: $Gr_{a,ex}^{S,\beta}(\mathbb{R}^3) \to Gr_a^{S,\beta}(\mathbb{R}^3)$

$(\Sigma, b_\Sigma) \mapsto (\Sigma, \delta\beta_\Sigma)$

is a principal $\mathrm{Diff}(S,\beta)/\mathrm{Diff}_{ex}(S,\beta) \simeq \mathbb{T}_\ell$ bundle.

Onsager - Feynman prequantization condition

$$\boxed{a\ell \in 2\pi\mathbb{Z}}$$

The 1-form $\theta = \widehat{\nu\cdot\beta}$ on $\mathrm{Emb}_a(S, \mathbb{R}^3)$ is $\mathrm{Diff}_{ex}(S,\beta)$-basic,

so it descends to $Gr^{S,\beta}_{a,ex}(\mathbb{R}^3)$

But it reproduces infinitesimal generators of $\mathbb{T}_\ell$-action

only up to the factor a

$$\zeta_\lambda \mapsto a\lambda$$

Prequantum bundle

$$Gr^{S,\beta}_{a,ex}(\mathbb{R}^3)/K \longrightarrow Gr^{S,\beta}_a(\mathbb{R}^3)$$ has structure group $\mathbb{T}_\ell/K \simeq \mathbb{T}_{2\pi}$

where $K = \mathrm{Ker}(m_a : \mathbb{T}_\ell \to \mathbb{T}_{2\pi}) \simeq \mathbb{Z}_{\underbrace{\frac{a\ell}{2\pi}}_{\in\mathbb{Z}}} \simeq \mathbb{T}_{2\pi}$ and $\zeta_\lambda \mapsto \lambda$.

Polarization subgroup

$$G_{(\Sigma,\beta_\Sigma)} \subset G_\Sigma \subset G$$

isotropy sgr. $\qquad$ $\overset{\text{"not}}{H}$ polarization sgr. with $\mathfrak{h} = \mathfrak{g}_\Sigma$

$(\Sigma, \beta_\Sigma) \in \mathfrak{g}^*$ vanishes on $[\mathfrak{h}, \mathfrak{h}]$

Lie alg. homo. $(\Sigma, \beta_\Sigma) : \mathfrak{h} \to \mathbb{R}$

can be integrated to a group homomorphism

$$\chi : H \to \mathbb{T}_{2\pi}$$

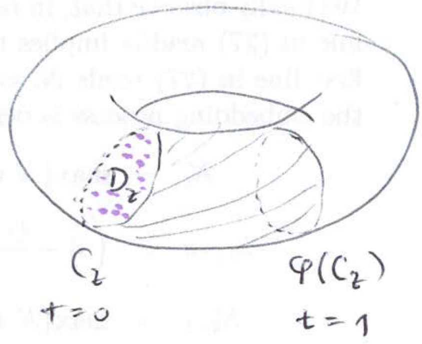
i.e. a character, iff $a\ell \in 2\pi\mathbb{Z}$.

Rem (Goldin, Menikoff, Sharp): Vortex filaments in 3D have no polarization.

Character - geometrically

- φ_t isotopy in H from id_Σ to φ

- $C_z^{\varphi_t}$ 2-chain swept out by C_z under φ_t



It stays on Σ!

- choose $\partial D_z = C_z$ and let $\check{D}_z^{\varphi_t}$ 3-chain

Special case: $D_z^{\varphi_t}$ inside Σ

$$\chi(\varphi) = \int_0^\ell \Big(\int_{D_z^{\varphi_t}} \underbrace{\tilde{\mu}}_{d\nu} \Big) dz \mod 2\pi\mathbb{Z} \quad \in \mathbb{T}_{2\pi}$$

well def. group homomorphism since $[C_z^{\varphi_t}] = k[\Sigma]$, $k \in \mathbb{Z}$

$$\chi(\text{loop}) = \int_0^\ell \Big(\int_{C_z^{\varphi_t}} \nu \Big) dz = k \int_0^\ell \Big(\int_\Sigma \nu \Big) dz = k \underbrace{a\ell}_{\in 2\pi\mathbb{Z}} \in 2\pi\mathbb{Z} \quad \square$$

Thanks for being here!