Seiberg–Witten Theory

An Introduction

Friday Fish Seminar
Utrecht University

2025

Contents

Lecture 2

Clifford algebras and spin groups

2.1 Clifford algebras

2.1.1 Basic definitions and properties

Definition 2.1.1. Let $(V, \langle\cdot,\cdot\rangle)$ be a Euclidean vector space. The **Clifford algebra** of V is defined as

$$\mathrm{Cl}(V) := T(V)/I(V),$$

where $T(V) := \bigoplus_{r\geq 0} V^{\otimes r}$ is the tensor algebra of V and $I(V)$ is the two-sided ideal generated by $\{v \otimes v + \|v\|^2 : v \in V\}$.

$\mathrm{Cl}(V)$ is a unital associative real algebra whose product we will denote by ab or $a \cdot b$.

Remark 2.1.2. $I(V)$ is also generated by $\{v \otimes w + w \otimes v + 2\langle v, w\rangle : v, w \in V\}$.

Choosing an orthonormal basis $(e_i)_i$ for V we can describe $\mathrm{Cl}(V)$ as the algebra generated by $\{e_i\}_i$ subject to the relations

$$e_i e_j = \begin{cases} -1, & i = j, \\ -e_j e_i, & i \neq j. \end{cases}$$

In particular, $\dim \mathrm{Cl}(V) = 2^{\dim V}$.

Lemma 2.1.3. *The map $V \to \mathrm{Cl}(V)$ given by the composition $V \hookrightarrow T(V) \to \mathrm{Cl}(V)$ is injective.*

Proof. It is enough to prove that $I(V) \cap V = 0$. Let

$$a = \sum_i b_i \otimes (v_i \otimes v_i + \|v_i\|^2) \otimes c_i \in I(V),$$

where the sum is finite and $b_i, c_i \in \mathrm{Cl}(V)$ are homogeneous. If $a \in V$, then necessarily

$$\sum_j b_j \otimes v_j \otimes v_j \otimes c_j = 0,$$

where j ranges along the indices such that $|b_j| + |c_j|$ is maximal. Then it also follows that

$$\sum_j b_j \|v_j\|^2 \otimes c_j = 0.$$

We now proceed by induction to conclude that $a = 0$. □

Proposition 2.1.4. $\mathrm{Cl}(V)$ *is, up to isomorphism, the unique unital associative real algebra with an inclusion* $V \hookrightarrow \mathrm{Cl}(V)$ *satisfying the following universal property: any linear map* $f : V \to A$ *to a unital associative real algebra* A *such that* $f(v)^2 + \|v\|^2 = 0$ *extends uniquely to an algebra morphism* $\mathrm{Cl}(V) \to A$*.*

Corollary 2.1.5. *An isometry* $(V, \langle\cdot,\cdot\rangle) \to (W, \langle\cdot,\cdot\rangle)$ *induces a unique algebra morphism* $\mathrm{Cl}(V) \to \mathrm{Cl}(W)$*. In particular,* $\mathrm{O}(V) \subseteq \mathrm{Aut}(\mathrm{Cl}(V))$*.*

$\mathrm{Cl}(V)$ is actually a $\mathbb{Z}_2$-graded algebra, or super algebra.

Definition 2.1.6. We denote by ϵ the algebra automorphism of $\mathrm{Cl}(V)$ induced by the antipodal map $V \to V$ sending $v \in V$ to $-v$.

Lemma 2.1.7. *We have that* $\epsilon^2 = 1$*, so* ϵ *induces a* $\mathbb{Z}_2$*-grading* $\mathrm{Cl}(V) = \mathrm{Cl}^0(V) \oplus \mathrm{Cl}^1(V)$ *with respect to which* $\mathrm{Cl}(V)$ *becomes a graded algebra.*

Definition 2.1.8. We call $\mathrm{Cl}^0(V)$ the **even part** of $\mathrm{Cl}(V)$ and $\mathrm{Cl}^1(V)$ the **odd part**.

Example 2.1.9. Let $\mathrm{Cl}(n)$ be the Clifford algebra of n-dimensional Euclidean space. Then

1. $\mathrm{Cl}(1) = \mathbb{R}[x]/(x^2+1) \cong \mathbb{C}$.
2. $\mathrm{Cl}(2)$ is generated by x and y subject to $x^2 = y^2 = -1$ and $xy = -yx$, i.e., $\mathrm{Cl}(2) \cong \mathbb{H}$. On the other hand, $\mathrm{Cl}^0(2) = \mathbb{R}[xy]/((xy)^2 - 1) \cong \mathbb{C}$.
3. $\mathrm{Cl}(3) \cong \mathbb{H} \oplus \mathbb{H}$ (exercise).

$\mathrm{Cl}(V)$ also comes equipped with a transposition operator.

Definition 2.1.10. We define the **transposition** map $^t : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ as the induced map on $\mathrm{Cl}(V)$ by the transposition map on $T(V)$ given by

$$v_1 \otimes \cdots \otimes v_k \longmapsto v_k \otimes \cdots \otimes v_1.$$

The transposition map satisfies the properties that one would expect: $(a^t)^t = a$ and $(ab)^t = b^t a^t$.

We now want to take a closer look at the relationship between $\mathrm{Cl}(V)$ and $\wedge V$. Observe that $\dim \mathrm{Cl}(V) = \dim \wedge V$, so that, abstractly, we have that $\mathrm{Cl}(V) \cong \wedge V$ noncanonically as ungraded vector spaces. We will see now that there is actually a canonical isomorphism of graded vector spaces $\mathrm{Cl}(V) \cong \wedge V$.

Definition 2.1.11. A **representation** or **Clifford module** of $\mathrm{Cl}(V)$ is a real vector space S together with an algebra morphism $\gamma : \mathrm{Cl}(V) \to \mathrm{End}(S)$, called the **Clifford action**, which we will often write as $\gamma_a \psi$ or $a \cdot \psi$, for $a \in \mathrm{Cl}(V)$ and $\psi \in S$.

Example 2.1.12.

1. $\mathrm{Cl}(V)$ is a Clifford module via left multiplication.
2. If S is a Clifford module with Clifford action γ, then S^* is also a Clifford module with Clifford action γ^* given by $(\gamma^*)_a := (\gamma_{a^t})^*$.

Lemma 2.1.13. *The map* $V \to \mathrm{End}(\wedge V)$ *sending* $v \in V$ *to* $\psi \mapsto v \wedge \psi - i_v \psi$*, where*

$$i_v(v_1 \wedge \cdots \wedge v_k) := \sum_i (-1)^{i+1} \langle v, v_i \rangle v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_k,$$

induces a representation of $\mathrm{Cl}(V)$ *on* $\wedge V$*.*

Proof. For $v \in V$ and $\psi \in \wedge V$,

$$v \cdot (v \cdot \psi) = v \cdot (v \wedge \psi - i_v \psi) = -i_v(v \wedge \psi) - v \wedge i_v \psi = -\|v\|^2 \psi. \qquad \square$$

Definition 2.1.14. We define the **symbol map** $\sigma : \mathrm{Cl}(V) \to \wedge V$ by $\sigma(a) := a \cdot 1$, where a acts on $1 \in \mathbb{R} = \wedge^0 V$ by the representation from Lemma 2.1.13.

In low degrees, σ looks like:

$$\begin{aligned} \sigma(1) &= 1, \\ \sigma(v) &= v, \\ \sigma(v_1 v_2) &= v_1 \wedge v_2 - \langle v_1, v_2 \rangle, \\ \sigma(v_1 v_2 v_3) &= v_1 \wedge v_2 \wedge v_3 - \langle v_1, v_2 \rangle v_3 + \langle v_1, v_3 \rangle v_2 - \langle v_2, v_3 \rangle v_1 \end{aligned}$$

Proposition 2.1.15. *The symbol map is an isomorphism of graded vector spaces.*

Proof. If $(e_i)_i$ is an orthonormal basis for V, then

$$\sigma(e_{i_1} \dots e_{i_k}) = e_{i_1} \wedge \cdots \wedge e_{i_k}, \qquad \text{for } i_1 < \cdots < i_k,$$

so that σ sends a basis for $\mathrm{Cl}(V)$ to a basis for $\wedge V$. $\square$

Remark 2.1.16. The inverse of the symbol map $q : \wedge V \to \mathrm{Cl}(V)$ is sometimes called the **quantization map**.

Lastly, we will need at some point how Clifford algebras behave under direct sums.

Proposition 2.1.17. *If* $V = W_1 \oplus W_2$ *is an orthogonal decomposition, then* $\mathrm{Cl}(V) \cong \mathrm{Cl}(W_1) \otimes \mathrm{Cl}(W_2)$ *as graded algebras.*

Proof. Consider the linear map $f : V \to \mathrm{Cl}(W_1) \otimes \mathrm{Cl}(W_2)$ given by $f(w_1 + w_2) := w_1 \otimes 1 + 1 \otimes w_2$. Then

$$\begin{aligned} f(w_1 + w_2) f(w_1 + w_2) &= (w_1 \otimes 1 + 1 \otimes w_2)^2 \\ &= w_1^2 \otimes 1 + w_1 \otimes w_2 - w_1 \otimes w_2 + 1 \otimes w_2^2 \\ &= -\|w_1 + w_2\|^2. \end{aligned}$$

It lifts, then, to a linear map $\mathrm{Cl}(V) \to \mathrm{Cl}(W_1) \otimes \mathrm{Cl}(W_2)$. One can see that this map is surjective by taking an orthonormal basis of V adapted to the decomposition $W_1 \oplus W_2$. Since the dimensions agree, it must be an isomorphism. $\square$

2.1.2 Chirality

Definition 2.1.18. Let $\mathrm{vol} \in \det V$ be a volume element normalized such that $\|\mathrm{vol}\|^2 = 1$. Then we define the corresponding **chirality element** $\Gamma := q(\mathrm{vol}) \in \mathrm{Cl}(V)$.

Lemma 2.1.19. $\Gamma^2 = (-1)^{n(n+1)/2}$ *and* $\Gamma v = (-1)^{n-1} v \Gamma$, *where* $n = \dim V$. *In particular,* Γ *is in the center of* $\mathrm{Cl}(V)$ *if* n *is odd.*

Proof. Let $(e_i)_i$ be an orthonormal basis for V. Then $\Gamma = e_1 \dots e_n$, so

$$\Gamma^2 = e_1 \dots e_n e_1 \dots e_n = (-1)^{n-1+n-2+\dots+1} e_1^2 \dots e_n^2 = (-1)^{n(n+1)/2}.$$

On the other hand, if $v \neq 0$, then let $e_1 = v/\|v\|$, and hence

$$v\Gamma = \|v\| e_1 e_1 \dots e_n = (-1)^{n-1} \|v\| e_1 \dots e_n e_1 = (-1)^{n-1} \Gamma v. \qquad \square$$

Recall that given a volume element vol $\in \det V$ we can define its **Hodge star** operator $* : \wedge^k V \to \wedge^{n-k} V$, where $n = \dim V$, by $*w := i_w \mathrm{vol}$, where we use the metric to identify w with an element in $\wedge^k V^*$. Alternatively, $*w$ is the unique element in $\wedge^{n-k} V$ such that $u \wedge *w = \langle u, w \rangle \mathrm{vol}$ for all $u \in \wedge^k V$.

Proposition 2.1.20. *Let* vol $\in \det V$ *be a normalized volume element. Then its chirality element and its Hodge star are related by* $\sigma(a\Gamma) = *\sigma(\epsilon(a^t))$.

Proof. First of all, notice that if $a, b \in \mathrm{Cl}(V)$, then $\sigma(ab) = ab \cdot 1 = a \cdot (b \cdot 1) = a \cdot \sigma(b)$. Applying this to $b = \Gamma$ we get that $\sigma(a\Gamma) = a \cdot \mathrm{vol}$. Let $(e_i)_i$ be an orthonormal basis such that $\mathrm{vol} = e_1 \wedge \dots \wedge e_n$. Then

$$\begin{aligned} e_{i_1} \dots e_{i_k} \cdot \mathrm{vol} &= -e_{i_1} \dots e_{i_{k-1}} \cdot i_{e_{i_k}} (e_1 \wedge \dots \wedge e_n) \\ &= e_{i_1} \dots e_{i_{k-2}} \cdot i_{e_{i_{k-1}}} i_{e_{i_k}} (e_1 \wedge \dots \wedge e_n) \\ &= \dots = (-1)^k i_{e_{i_1}} \dots i_{e_{i_k}} \mathrm{vol} = (-1)^k i_{e_{i_k} \wedge \dots \wedge e_{i_1}} \mathrm{vol} \\ &= (-1)^k * (e_{i_k} \wedge \dots \wedge e_{i_1}). \square \end{aligned}$$

When working with the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ we can further normalize our chirality elements. We consider $V_{\mathbb{C}}$ endowed with the extension of $\langle \cdot, \cdot \rangle$ to $V_{\mathbb{C}}$ by $\mathbb{C}$-bilinearity.

Definition 2.1.21. We define $\mathbb{C}\mathrm{l}(V) := \mathrm{Cl}(V_{\mathbb{C}})$. If $\dim V = 2n$ and $\Gamma \in \mathrm{Cl}(V)$ is a chirality element, we define the **complex chirality element** $\Gamma_c := i^n \Gamma \in \mathbb{C}\mathrm{l}(V)$.

Remark 2.1.22. $\mathbb{C}\mathrm{l}(V) \cong \mathrm{Cl}(V) \otimes_{\mathbb{R}} \mathbb{C}$ canonically.

Proposition 2.1.23. *Let* $\dim V = 2n$. *Then the complex chirality element satisfies* $\Gamma_c^2 = 1$, *so it induces a decomposition* $\mathbb{C}\mathrm{l}(V) = \mathbb{C}\mathrm{l}_+(V) \oplus \mathbb{C}\mathrm{l}_-(V)$. *Moreover, if* $n = 2$, *then the symbol map induces an isomorphism*

$$\mathbb{C}\mathrm{l}^0_{\pm}(V) \cong \mathbb{C}(1 \pm \Gamma_c) \oplus \wedge^2_{\pm} V_{\mathbb{C}}.$$

Proof. First, by Lemma 2.1.19, $\Gamma_c^2 = i^{2n} \Gamma^2 = (-1)^n (-1)^{n(2n+1)} = 1$. By Proposition 2.1.20 we have that $a \in \mathbb{C}\mathrm{l}^0_{\pm}(V)$ if and only if

$$\pm\sigma(a) = i^n \sigma(a\Gamma) = i^n * \sigma(a^t) = i^n * \sigma(a)^t.$$

Hence, the symbol map identifies $\mathbb{C}\mathrm{l}^0_{\pm}(V)$ with the space

$$\{w \in \wedge^{\mathrm{even}} V_{\mathbb{C}} : i^n * w^t = \pm w\}.$$

If $n = 2$, then such a $w \in \wedge^{\mathrm{even}} V_{\mathbb{C}}$ in this space can be decomposed as $w = w_0 + w_2 + w_4$, for $w_j \in \wedge^j V$, and

$$i^n * w^t = -*w_0 + *w_2 - *w_4 = \pm w_0 \pm w_2 \pm w_4.$$

Hence, $*w_0 = \mp w_4$, $*w_2 = \pm w_2$ and $*w_4 = \mp w_0$. $\square$

2.2 Spin and spin^c groups

2.2.1 Spin groups

Definition 2.2.1. We define the group of **units** of $\mathrm{Cl}(V)$ as the group of invertible elements

$$\mathrm{Cl}^\times(V) := \{a \in \mathrm{Cl}(V) : \text{there is } a^{-1} \in \mathrm{Cl}(V) \text{ with } aa^{-1} = a^{-1}a = 1\}.$$

We define the **adjoint** action as the map $\mathrm{Ad} : \mathrm{Cl}^\times(V) \to \mathrm{Aut}(\mathrm{Cl}(V))$ given by

$$\mathrm{Ad}_a b := \epsilon(a)ba^{-1}.$$

Remark 2.2.2. $\mathrm{Cl}(V)$ has a unique smooth structure (the vector space smooth structure) making the multiplication. $\mathrm{Cl}^\times(V)$ is an open subspace thereof, so that $\mathrm{Cl}^\times(V)$ becomes a Lie group. With this structure, $\mathrm{Ad} : \mathrm{Cl}^\times(V) \to \mathrm{Aut}(\mathrm{Cl}(V))$ is a Lie group map.

The adjoint action by vectors in V has a familiar form.

Proposition 2.2.3. *Let $v \in V$ be nonzero. Then $\mathrm{Ad}_v V = V$ and Ad_v acts on V as the reflection along the hyperplane $v^\perp$, that is:*

$$\mathrm{Ad}_v w = w - 2\frac{\langle v, w\rangle}{\|v\|^2}v, \qquad \textit{for } w \in V.$$

Proof. Since v is nonzero, $v^{-1} = -v/\|v\|^2$. Then

$$\mathrm{Ad}_v w = -vwv^{-1} = w - 2\frac{\langle v, w\rangle}{\|v\|^2}v.$$

□

Definition 2.2.4. We define the **Clifford group** of V as

$$\Gamma(V) := \{a \in \mathrm{Cl}^\times(V) : \mathrm{Ad}_a V = V\}.$$

Remark 2.2.5. $\Gamma(V)$ is a closed subgroup of $\mathrm{Cl}^\times(V)$, so that it becomes an embedded Lie subgroup of $\mathrm{Cl}^\times(V)$ by Cartan's closed subgroup theorem.

Proposition 2.2.6. *The kernel of $\mathrm{Ad} : \Gamma(V) \to \mathrm{GL}(V)$ is $\mathbb{R}^\times$.*

Proof. Let $a \in \Gamma(V)$ with $\mathrm{Ad}_a = 1$ and write $a = a_0 + a_1$, with a_0 even and a_1 odd. The fact that $\mathrm{Ad}_a = 1$ means that $va_0 + va_1 = a_0v - a_1v$ for all $v \in V$. Let $(e_i)_i$ be an orthonormal basis for V. Then a_0 and a_1 are polynomials on $(e_i)_i$, which we can write as $a_0 = e_1b_1 + c_0$ and $a_1 = e_1b_0 + c_1$, for b_1 and c_1 odd polynomials on $(e_i)_{i>1}$ and b_0 and c_0 even polynomials on $(e_i)_{i>1}$. Then

$$\begin{aligned} e_1a &= e_1a_0 + e_1a_1 = e_1^2b_1 + e_1c_0 + e_1^2b_0 + e_1c_1 = -b_1 - b_0 + e_1c_0 + e_1c_1 \\ &= a_0e_1 - a_1e_1 = e_1b_1e_1 + c_0e_1 - e_1b_0e_1 - c_1e_1 = -e_1^2b_1 + e_1c_0 - e_1^2b_0 + e_1c_1 \\ &= b_1 + b_0 + e_1c_0 + e_1c_1, \end{aligned}$$

which implies that $b_0 = b_1 = 0$. Hence, $a = c_0 + c_1$ is a polynomial on $(e_i)_{i>1}$. By induction, we see that actually $a \in \mathbb{R}$, and since $a \in \mathrm{Cl}^\times(V)$, then $a \in \mathbb{R}^\times$. □

Definition 2.2.7. We define the **norm** map $N : \mathrm{Cl}(V) \to \mathrm{Cl}(V)$ by $N(a) := \epsilon(a^t)a$.

Proposition 2.2.8. *The restriction of the norm to $\Gamma(V)$ gives a Lie group morphism*

$$N : \Gamma(V) \to \mathbb{R}^\times.$$

Proof. Let $a \in \Gamma(V)$. Then for all $v \in V$ we have that

$$\mathrm{Ad}_a v = \epsilon(a)va^{-1} = (\mathrm{Ad}_a v)^t = \epsilon(a^{-t})va^t,$$

so $\epsilon(a^t)av(\epsilon(a^t)a)^{-1} = v$, i.e., $N(a)$ lies in the kernel of Ad. The same expression gives that $\mathrm{Ad}_{a^t} = \mathrm{Ad}_{a^{-1}}$, so that $a^t \in \Gamma(V)$, and therefore $N(a) \in \Gamma(V)$, and by Proposition 2.2.6 we now conclude that $N(a) \in \mathbb{R}^\times$. On the other hand, if $a, b \in \Gamma(V)$, then

$$N(ab) = \epsilon(b^t a^t)ab = \epsilon(b^t)N(a)b = N(a)N(b). \qquad \square$$

Corollary 2.2.9. *The restriction of* Ad *to* $\Gamma(V)$ *lands in* $\mathrm{O}(V)$*, i.e.,* $\mathrm{Ad} : \Gamma(V) \to \mathrm{O}(V)$*.*

Proof. Let $a \in \Gamma(V)$. First note that $N(\epsilon(a)) = N(a)$ and that $v \in \Gamma(V)$ for every nonzero $v \in V$. Then

$$\|\mathrm{Ad}_a v\|^2 = N(\mathrm{Ad}_a v) = N(\epsilon(a)va^{-1}) = N(a)N(v)N(a^{-1}) = N(v) = \|v\|^2. \qquad \square$$

Definition 2.2.10. We define the **pin** and **spin** groups, respectively, of V as

$$\mathrm{Pin}(V) := \{a \in \Gamma(V) : N(a) = 1\},$$
$$\mathrm{Spin}(V) := \mathrm{Pin}(V) \cap \mathrm{Cl}^0(V).$$

Remark 2.2.11. Since $\mathrm{Pin}(V)$ is the kernel of a Lie group map, it is itself an embedded Lie subgroup of $\Gamma(V)$. Since $\mathrm{Cl}^0(V)$ is a closed subspace of $\mathrm{Cl}(V)$, then $\mathrm{Spin}(V)$ is an embedded Lie subgroup of $\mathrm{Pin}(V)$.

Theorem 2.2.12. *There is a short exact sequence*

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \Gamma(V) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V) \longrightarrow 1,$$

which restricts to

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}(V) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V) \longrightarrow 1.$$

In particular, $\mathrm{Spin}(V)$ *is the universal cover of* $\mathrm{SO}(V)$ *if* $\dim V \geq 3$*.*

Proof. Recall the Cartan–Dieudonné theorem: every element of $\mathrm{O}(V)$ can be written as a composition of at most $\dim V$ reflections. Together with Proposition 2.2.3, this gives surjectivity of Ad. That the kernel of Ad restricted to $\Gamma(V)$ is $\mathbb{R}^\times$ is exactly the content of Proposition 2.2.6. Moreover, if $a \in \mathrm{Pin}(V)$ is in the kernel of Ad, then $a \in \mathbb{R}^\times$, and since $N(a) = a^2 = 1$, then $a = \pm 1$.

Finally, the long exact sequence on homotopy groups gives that the index of $\pi_1\mathrm{Spin}(V)$ in $\pi_1\mathrm{SO}(V)$ is 2. Since $\pi_1\mathrm{SO}(V) = \mathbb{Z}_2$ if $\dim V \geq 3$, then $\pi_1\mathrm{Spin}(V) = 1$ in this case. $\square$

Corollary 2.2.13. *We have that*

$$\Gamma(V) = \{v_1 \dots v_r \in \mathrm{Cl}^\times(V) : v_i \in V, \|v_i\|^2 \neq 0, r \geq 0\},$$
$$\mathrm{Pin}(V) = \{v_1 \dots v_r \in \mathrm{Cl}^\times(V) : v_i \in V, \|v_i\|^2 = 1, r \geq 0\}.$$

Proof. Clearly $v_1 \dots v_r \in \Gamma(V)$ if $\|v_i\|^2 \neq 0$. To prove the converse, let $a \in \Gamma(V)$. Then Ad_a is a composition of at most $\dim V$ reflections, say along the nonzero vectors $v_1, \dots, v_r \in V$. Then $\mathrm{Ad}_a = \mathrm{Ad}_{v_1 \dots v_r}$, so by Theorem 2.2.12 we have that $a = \lambda v_1 \dots v_r$, for some $\lambda \in \mathbb{R}^\times$. A similar argument gives the result for $\mathrm{Pin}(V)$ as well. □

Corollary 2.2.14. *The dimension of* $\mathrm{Spin}(V)$ *as a Lie group is* $\frac{1}{2}\dim V(\dim V - 1)$, *and its Lie algebra is*

$$\mathfrak{so}(V) = \{A \in \mathfrak{gl}(V) : \langle A\cdot, \cdot\rangle + \langle \cdot, A\cdot\rangle = 0\}.$$

2.2.2 Spin^c groups

The same proofs as in Theorem 2.2.12 and Proposition 2.2.8 give the short exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma(V_\mathbb{C}) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V_\mathbb{C}) \longrightarrow 1. \tag{2.1}$$

and that $N : \Gamma(V_\mathbb{C}) \to \mathbb{C}^\times$ is a Lie group morphism.

Definition 2.2.15. We define the **adjoint** map $^* : \mathbb{C}\mathrm{l}(V) \to \mathbb{C}\mathrm{l}(V)$ by $a^* := \overline{a}^t$. We also define the **norm**^{c} map $N^c : \mathbb{C}\mathrm{l}(V) \to \mathbb{C}\mathrm{l}(V)$ as $N^c(a) := \epsilon(a^*)a$.

The adjoint map satisfies, as expected, $(a^*)^* = a$, $(ab)^* = b^*a^*$ and $(\lambda a)^* = \overline{\lambda}a^*$ for $\lambda \in \mathbb{C}$. Notice as well that $N^c : \Gamma(V_\mathbb{C}) \to \Gamma(V_\mathbb{C})$ is a Lie group map, which we do not know to necessarily land in $\mathbb{C}^\times$.

Lemma 2.2.16. *An element* $a \in \Gamma(V_\mathbb{C})$ *satisfies* $\overline{\mathrm{Ad}_a v} = \mathrm{Ad}_a \overline{v}$ *for all* $v \in V_\mathbb{C}$ *if and only if* $N^c(a) \in \mathbb{R}^\times$.

Proof. $\overline{\mathrm{Ad}_a v} = \mathrm{Ad}_a \overline{v}$ if and only if $(\mathrm{Ad}_a v)^* = \mathrm{Ad}_a v^*$, which is equivalent to $\epsilon(a^*)a \in \mathbb{C}^\times$ by (2.1). In such case, we have that

$$\overline{\epsilon(a^*)a} = (\epsilon(a^*)a)^* = \epsilon(a^*)a,$$

so that actually $N^c(a) \in \mathbb{R}^\times$. □

Definition 2.2.17. We define the **Clifford**^{c}, **pin**^{c} and **spin**^{c} groups as

$$\begin{aligned} \Gamma^c(V) &:= \{a \in \Gamma(V_\mathbb{C}) : N^c(a) \in \mathbb{R}^\times\}, \\ \mathrm{Pin}^c(V) &:= \{a \in \Gamma^c(V) : N^c(a) = 1\}, \\ \mathrm{Spin}^c(V) &:= \mathrm{Pin}^c(V) \cap \mathbb{C}\mathrm{l}^0(V). \end{aligned}$$

Remark 2.2.18. Since $\mathbb{R}^\times$ is a closed Lie subgroup of $\Gamma(V_\mathbb{C})$, then $\Gamma^c(V)$ is also a closed Lie subgroup of $\Gamma(V_\mathbb{C})$. Then $N^c : \Gamma^c(V) \to \mathbb{R}^\times$ is a Lie group map, so that $\mathrm{Pin}^c(V)$ becomes a closed Lie subgroup of $\Gamma^c(V)$, and hence $\mathrm{Spin}^c(V)$ a closed Lie subgroup of $\mathrm{Pin}^c(V)$.

Theorem 2.2.19. *There is a short exact sequence*

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow \Gamma^c(V) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V) \longrightarrow 1,$$

which restricts to

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Pin}^c(V) \xrightarrow{\mathrm{Ad}} \mathrm{O}(V) \longrightarrow 1,$$

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\mathrm{Ad}} \mathrm{SO}(V) \longrightarrow 1.$$

Moreover, there is also a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \Gamma^c(V) \xrightarrow{\mathrm{Ad}\times N} \mathrm{O}(V) \times \mathbb{C}^\times \longrightarrow 1,$$

which restricts to

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Pin}^c(V) \xrightarrow{\mathrm{Ad}\times N} \mathrm{O}(V) \times \mathrm{U}(1) \longrightarrow 1,$$
$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\mathrm{Ad}\times N} \mathrm{SO}(V) \times \mathrm{U}(1) \longrightarrow 1.$$

Proof. The first three are clear. The second one follows from the fact that if $(\mathrm{Ad}_a, N(a)) = (1,1)$, then $a \in \mathbb{C}^\times$ and $a^2 = 1$, so $a = \pm 1$. For the last two, notice that if $a \in \mathrm{Pin}^c(V)$, then $N^c(a) = \epsilon(a^*)a = \epsilon(\overline{a}^t)a = 1$, so that $N(a) = \epsilon(a^t)a = \overline{a}^{-1}a \in \mathbb{C}^\times$, and

$$N(a)\overline{N(a)} = \overline{a}^{-1}aa^{-1}\overline{a} = 1,$$

so $N(a) \in \mathrm{U}(1)$. ☐

Corollary 2.2.20. *We have that*

$$\Gamma^c(V) = \{v_1 \dots v_r \in \mathbb{C}\mathrm{l}^\times(V) : v_i \in V_\mathbb{C}, \langle v_i, v_i\rangle \neq 0, r \geq 0\},$$
$$\mathrm{Pin}^c(V) = \{v_1 \dots v_r \in \mathbb{C}\mathrm{l}^\times(V) : v_i \in V_\mathbb{C}, \langle v_i, v_i\rangle \in \mathrm{U}(1), r \geq 0\}.$$

Moreover, the group morphisms

$$\Gamma(V) \times \mathbb{C}^\times \longrightarrow \Gamma^c(V),$$
$$\mathrm{Pin}(V) \times \mathrm{U}(1) \longrightarrow \mathrm{Pin}^c(V)$$
$$\mathrm{Spin}(V) \times \mathrm{U}(1) \longrightarrow \mathrm{Spin}^c(V)$$

given by $(a, z) \mapsto az$ induce group isomorphisms

$$\Gamma^c(V) \cong \Gamma(V) \times_{\mathbb{R}^\times} \mathbb{C}^\times,$$
$$\mathrm{Pin}^c(V) \cong \mathrm{Pin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1),$$
$$\mathrm{Spin}^c(V) \cong \mathrm{Spin}(V) \times_{\mathbb{Z}_2} \mathrm{U}(1).$$

Proof. The claims for $\Gamma^c(V)$ and $\mathrm{Pin}^c(V)$ are clear by Theorem 2.2.19.

If $a \in \Gamma(V)$ and $z \in \mathbb{C}^\times$ are such that $az = 1$, then $a|z|^2 = \overline{z}$, and since a is real, we get that $z \in \mathbb{C}^\times \cap \mathbb{R} = \mathbb{R}^\times$. On the other hand, if $a \in \Gamma^c(V)$, then there are vectors $v_i \in V$ with $\|v_i\|^2 \neq 0$ such that $\mathrm{Ad}_a = \mathrm{Ad}_{v_1 \dots v_r}$. Theorem 2.2.19 now implies that there is $z \in \mathbb{C}^\times$ such that $a = zv_1 \dots v_r$. Similar reasonings apply to $\mathrm{Pin}^c(V)$ and $\mathrm{Spin}^c(V)$. ☐

2.3 Spin representations

Let $\dim V = 2n$. We will be concerned with complex, and even unitary, spin representations.

Definition 2.3.1. A Clifford module S with action $\gamma : \mathbb{C}\mathrm{l}(V) \to \mathrm{End}(S)$ is **unitary** if S carries a Hermitian metric such that $\gamma_{\epsilon(a^*)} = \gamma_a^*$.

Equivalently, a representation of $\mathbb{C}\mathrm{l}(V)$ is unitary if and only if every real vector $v \in V \subseteq V_{\mathbb{C}}$ acts as a skew-self-adjoint operator.

Definition 2.3.2. We define the **trace** on $\mathbb{C}\mathrm{l}(V)$ as the linear map $\mathrm{tr} : \mathbb{C}\mathrm{l}(V) \to \mathbb{C}$ given by $\mathrm{tr}(a) := \sigma(a)_0$, where $_0$ denotes de degree 0 part.

Lemma 2.3.3. *The trace satisfies the following properties:*

$$\mathrm{tr}(ab) = \mathrm{tr}(ba), \qquad \mathrm{tr}(a^*) = \overline{\mathrm{tr}(a)}, \qquad \mathrm{tr}(1) = 1.$$

Example 2.3.4.

1. Let h be the Hermitian metric on $V_{\mathbb{C}}$ induced by $\langle \cdot, \cdot \rangle$, given by $h(v, w) := \langle \overline{v}, w \rangle$, which in turn induces a Hermitian metric on $\wedge V_{\mathbb{C}}$. Then the representation $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\wedge V_{\mathbb{C}})$ is unitary.

2. $\mathbb{C}\mathrm{l}(V)$ itself carries a Hermitian metric, given by $h(a, b) := \mathrm{tr}(\epsilon(a^*)b)$. Then the action of $\mathbb{C}\mathrm{l}(V)$ on itself by left multiplication becomes a unitary action. Moreover, with this Hermitian metric the symbol map $\sigma : \mathbb{C}\mathrm{l}(V) \to \wedge V_{\mathbb{C}}$ becomes an isometry:

$$\|\sigma(a)\|_h^2 = h(a \cdot 1, a \cdot 1) = h(1, \epsilon(a^*)a \cdot 1) = \mathrm{tr}(\epsilon(a^*)a).$$

Lemma 2.3.5. $(V_{\mathbb{C}}, \langle \cdot, \cdot \rangle)$ *admits Lagrangian subspaces. If L is a Lagrangian subspace, then it always admits a Lagrangian complement $L' \cong L^*$ inducing an isometry $V_{\mathbb{C}} \cong L \oplus L^*$, where the pairing on $L \oplus L^*$ is given by*

$$\langle v + \xi, w + \eta \rangle = \frac{1}{2}(\xi(w) + \eta(v)), \qquad \textit{for } v + \xi, w + \eta \in L \oplus L^*.$$

Proof. Let $(e_j)_j$ be an orthonormal basis for V and let $L := \mathrm{span}_{\mathbb{C}}\{e_j + ie_{n+j}\}_{j=1}^n$. Then $\dim_{\mathbb{C}} L = n = \frac{1}{2} \dim_{\mathbb{C}} V_{\mathbb{C}}$ and L is isotropic:

$$\langle e_j + ie_{n+j}, e_k + ie_{n+k} \rangle = \delta_{jk} - \delta_{jk} = 0.$$

Let L be now any Lagrangian and let W be any complement to it. Then from $V_{\mathbb{C}} = L \oplus W$ and $L \cap W = 0$ we get that $0 = L \cap W^\perp$ and $L \oplus W^\perp = V_{\mathbb{C}}$. Let $P : W \to L$ be the projection of W to L according to the decomposition $V_{\mathbb{C}} = L \oplus W^\perp$. Let $L' := \{w - \frac{1}{2}Pw : w \in W\}$. Since L' is the graph of a linear map $W \to L$, it is also a complement for L. Moreover, it is isotropic:

$$\langle w - \frac{1}{2}Pw, w - \frac{1}{2}Pw \rangle = \langle w, w - Pw \rangle + \frac{1}{4}\langle Pw, Pw \rangle = 0,$$

since $w - Pw \in W^\perp$ and $Pw \in L$. Hence, L' is a Lagrangian complement to L. Finally, $L' = (L')^\perp \cong (L')^\circ \cong L^*$. ☐

Proposition 2.3.6. *Let $L \subseteq V_{\mathbb{C}}$ be a Lagrangian and identify $V_{\mathbb{C}} \cong L \oplus L^*$. When endowed with the inherited Hermitian metric from $V_{\mathbb{C}}$, the space $\wedge L^*$ is a unitary Clifford module with Clifford action*

$$(v+\xi)\cdot\psi := \xi \wedge \psi - i_v\psi, \qquad \text{for } v+\xi \in L\oplus L^* \text{ and } \psi \in \wedge L^*,$$

whose dual Clifford module is $\wedge L$ with Clifford action

$$(v+\xi)\cdot w := v \wedge w - i_\xi w, \qquad \text{for } v+\xi \in L\oplus L^* \text{ and } w \in \wedge L.$$

Proof. It is indeed a Clifford module: if $v+\xi \in L\oplus L^*$ and $\psi \in \wedge L^*$, then

$$(v+\xi)^2\cdot\psi = (v+\xi)\cdot(\xi\wedge\psi - i_v\psi) = -i_v(\xi\wedge\psi) - \xi\wedge i_v\psi = -\langle v+\xi, v+\xi\rangle\psi.$$

To see that it is unitary, first notice that, via the identification $L^* \cong \overline{L}$, conjugation in $L\oplus L^*$ can be written as $\overline{v+\xi} = \overline{\xi} + \overline{v}$, with $\overline{\xi} \in L$ and $\overline{v} \in L^*$. Let $\psi = \alpha^1 \wedge \cdots \wedge \alpha^k$ and $\varphi = \beta^1 \wedge \cdots \wedge \beta^{k+1}$, for $\alpha^i, \beta^i \in L^*$. Then

$$\begin{aligned}
h((v+\xi)\cdot\psi,\varphi) &= h(\xi\wedge\alpha^1\wedge\cdots\wedge\alpha^k, \beta^1\wedge\cdots\wedge\beta^{k+1})\\
&= \det\begin{pmatrix} h(\xi,\beta^1) & \dots & h(\xi,\beta^{k+1})\\ h(\alpha^1,\beta^1) & \dots & h(\alpha^1,\beta^{k+1})\\ \vdots & & \vdots\\ h(\alpha^k,\beta^1) & \dots & h(\alpha^k,\beta^{k+1})\end{pmatrix}\\
&= \sum_i (-1)^{i+1} h(\xi,\beta^i)\det\begin{pmatrix} h(\alpha^1,\beta^1) & \dots & \widehat{h(\alpha^1,\beta^i)} & \dots & h(\alpha^1,\beta^{k+1})\\ \vdots & & \vdots & & \\ h(\alpha^k,\beta^1) & \dots & \widehat{h(\alpha^k,\beta^i)} & \dots & h(\alpha^k,\beta^{k+1})\end{pmatrix}\\
&= \sum_i (-1)^{i+1} h(\xi,\beta^i) h(\psi, \beta^1\wedge\cdots\wedge\widehat{\beta^i}\wedge\cdots\wedge\beta^{k+1})\\
&= h(\psi, i_{\overline{\xi}}\varphi) = h(\psi, \overline{(v+\xi)}\cdot\varphi).
\end{aligned}$$

That $\wedge L$ is a Clifford module is clear. It only remains to see that it is the Clifford dual to $\wedge L^*$. Consider the duality pairing $(\cdot,\cdot) : \wedge L^* \times \wedge L \to \mathbb{C}$ given by $(\psi, w) := i_w\psi$, then an analogous computation to the previous one proves that

$$((v+\xi)\cdot\psi, w) = (\psi, (v+\xi)\cdot w)\,.$$

Since this duality pairing is non-degenerate, it establishes that the Clifford dual of $\wedge L^*$ is indeed $\wedge L$. □

Definition 2.3.7. The $\mathbb{Cl}(V)$-module $S_L := \wedge L^*$, for a Lagrangian $L \subseteq V_{\mathbb{C}}$, we call the **spinor module**.

Theorem 2.3.8. *The spinor module S_L is irreducible and the Clifford action*

$$\mathbb{Cl}(V) \to \mathrm{End}(S_L)$$

is an isomorphism of graded algebras. It restricts to an isomorphism

$$\mathbb{Cl}^0(V) \to \mathrm{End}(S_L^0) \oplus \mathrm{End}(S_L^1),$$

so that both S_L^0 and S_L^1 are irreducible unitary $\mathbb{Cl}^0(V)$-modules, which are moreover non-isomorphic.

Proof. Note that the dimensions of $\mathrm{Cl}(L \oplus L^*)$ and $\mathrm{End}(\wedge L^*)$ agree, so to prove that $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(S_L)$ is an isomorphism it suffices to check that $\mathrm{End}(\wedge L^*)$ is generated by operators of the kind $\xi \wedge$ and i_v, for $\xi \in L^*$ and $v \in L$.

If $\dim L = 1$, let $v \in L$ be a generator for L and let $\xi \in L^*$ be such that $\xi(v) = 1$. Then $\wedge L^* = \mathbb{R} \oplus \mathbb{R}\xi$ and we can write the operators in matrix form

$$\xi \wedge \ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad i_v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad i_v \circ (\xi \wedge\ ) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Together with the identity matrix, these span $\mathrm{End}(\wedge L^*)$. For the general case, let $(v_i)_i$ be a basis for L with dual basis $(\xi^i)_i$, and let $L_i := \mathbb{R}v_i$, so that $L_i^* = \mathbb{R}\xi^i$. Then $L \oplus L^* = \bigoplus_i L_i \oplus L_i^*$ as quadratic vector spaces, so that, by the 1-dimensional case,

$$\mathrm{Cl}(L \oplus L^*) \cong \bigotimes_i \mathrm{Cl}(L_i \oplus L_i^*) \cong \bigotimes_i \mathrm{End}(\wedge L_i^*) = \mathrm{End}\left(\bigotimes_i \wedge L_i^*\right) = \mathrm{End}(\wedge L^*).$$

We conclude that indeed $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(S_L)$ is an isomorphism of algebras, which in particular implies that the representation is irreducible.

Restricting to the even parts, it induces an isomorphism

$$\mathbb{C}\mathrm{l}^0(V) \to \mathrm{End}(S_L^0) \oplus \mathrm{End}(S_L^1),$$

giving that both S_L^0 and S_L^1 are irreducible $\mathbb{C}\mathrm{l}^0(V)$-modules. To see that these are non-isomorphic, taking $(v_i)_i$ and $(\xi^i)_i$ dual bases for L and L^*, then consider the chirality element

$$\begin{aligned} \Gamma &:= (v_1 - \xi^1)(v_1 + \xi^1) \dots (v_n - \xi^n)(v_n + \xi^n) \\ &= (1 + 2v_1\xi^1) \dots (1 + 2v_n\xi^n) \end{aligned}$$

in $\mathrm{Cl}(L \oplus L^*)$. Since, for any multiindex I,

$$(1 + 2v_j\xi^j) \cdot \xi^I = \xi^I - 2i_{v_j}(\xi^j \wedge \xi^I) = \begin{cases} \xi^I, & j \in I \\ -\xi^I, & j \notin I, \end{cases}$$

we see that $\Gamma \cdot \xi^I = (-1)^n(-1)^{|\xi^I|}\xi^I$. Hence, Γ acts as $(-1)^n$ on S_L^0 and as $-(-1)^n$ on S_L^1, so they cannot be isomorphic. □

Theorem 2.3.9. *1. There is a unique isomorphism class of ungraded irreducible* $\mathbb{C}\mathrm{l}(V)$*-modules, represented by* S_L.

2. *There are two isomorphism classes of irreducible* $\mathbb{C}\mathrm{l}^0(V)$*-modules, represented by* S_L^0 *and* S_L^1.

3. *There are two isomorphism classes of graded irreducible* $\mathbb{C}\mathrm{l}(V)$*-modules, represented by* S_L *and* $S_L[1]$.

Proof. By Theorem 2.3.8, $\mathbb{C}\mathrm{l}(V)$ is isomorphic, as an ordinary algebra, to a complex matrix algebra, and these are known to have a unique irreducible representation. On the other hand, $\mathbb{C}\mathrm{l}^0(V)$ is isomorphic, as an ordinary algebra, to a direct sum of two complex matrix algebras, and again these are known to have two isomorphism classes of irreducible representations.

As for the graded case, it all boils down to checking how many different $\mathbb{Z}_2$-gradings does an ungraded spinor module S admit. We will see that any grading is given as the decomposition into eigenspaces for γ_{Γ_c}, where Γ_c is the complex chirality element and $\gamma : \mathbb{C}\mathrm{l}(V) \to \mathrm{End}(S)$ is the Clifford action. Indeed, since $\Gamma_c v = -v\Gamma_c$ for any $v \in V_{\mathbb{C}}$, then γ_v interchanges S^0 and S^1, so this $\mathbb{Z}_2$-grading is compatible with the action. Conversely, if $S = S^0 \oplus S^1$ is a compatible $\mathbb{Z}_2$-grading, since γ_v exchanges S^0 and S^1, for any $v \neq 0$, then both have dimension $\frac{1}{2}\dim S$. This implies that the restriction of the action $\mathbb{C}\mathrm{l}^0(V) \to \mathrm{End}(S^0) \oplus \mathrm{End}(S^1)$ is also an isomorphism, so that both S^0 and S^1 are irreducible $\mathbb{C}\mathrm{l}^0(V)$-modules. Since Γ_c lies in the center of $\mathbb{C}\mathrm{l}^0(V)$, then by Schur's lemma γ_{Γ_c} must act as a scalar on both S^0 and S^1, so these are its eigenspaces. □

Definition 2.3.10. The **(half-)spin representations** are the representations of $\mathrm{Spin}(V)$ induced by the two irreducible $\mathbb{C}\mathrm{l}^0(V)$-modules, which we call $S^{\pm}$. The induced representations of $\mathrm{Spin}^c(V)$ are the **(half-)spin^c representations**.

If $V = \mathbb{R}^{2n}$, then the spin and spin^c representations are typically denoted by $\Delta^{\pm}_{2n}$.

Proposition 2.3.11. *The spin and spin^c representations $S^{\pm}$ are irreducible and non-isomorphic.*

Proof. This follows immediately from Theorem 2.3.8 and from the fact that both $\mathrm{Spin}(V)$ and $\mathrm{Spin}^c(V)$ generate $\mathbb{C}\mathrm{l}^0(V)$ as a complex algebra. □

Lecture 3

—

Lecture 4

Dirac operators and the Seiberg–Witten equations

foThroughout this lecture, (M, g) will be an orientable riemannian manifold, whose Levi-Civita connection we denote by ∇^g. $C^\infty(M)$ will always refer to complex-valued smooth maps on M, whereas $C^\infty(M, \mathbb{R})$ will denote real-valued ones.

4.1 Dirac operators and spinor bundles

4.1.1 Dirac operators

Definition 4.1.1. A real (resp. complex) **Dirac bundle** over M is a pair (S, γ), where $S \to M$ is a Euclidean (resp. Hermitian) vector bundle and $\gamma : \mathrm{Cl}(M, g) \to \mathrm{End}(S)$ (resp. $\gamma : \mathbb{C}\mathrm{l}(M, g) \to \mathrm{End}(S)$) is a morphism of algebra bundles such that $\gamma_{\epsilon(a^t)} = \gamma_a^*$ (resp. $\gamma_{\epsilon(a^*)} = \gamma_a^*$). If S is $\mathbb{Z}_2$-graded such that γ is a morphism of graded algebra bundles, then we say that the Dirac bundle is **graded**. The action of γ we write as $\gamma_a\psi$ or $a \cdot \psi$, for $a \in \mathrm{Cl}(M, g)$ and $\psi \in S$.

We would like to do differential geometry with these objects, so we would like to introduce connections which are well suited for Dirac bundles.

Lemma 4.1.2. *There is a unique connection ∇^g on $\mathrm{Cl}(M, g)$ extending the Levi-Civita connection on M such that*

$$\nabla^g_X(ab) = (\nabla^g_X a)b + a\nabla^g_X b.$$

Proof. ∇^g induces a connection on the tensor algebra bundle $T(M, g)$ compatible with the algebra structure. To see that it induces one on $\mathrm{Cl}(M, g)$ it is enough to check that the ideal bundle $I(M, g)$ is parallel. If $X, Y \in \mathfrak{X}(M)$, then

$$\nabla^g_X(Y \otimes Y + \|Y\|_g^2) = \nabla^g_X Y \otimes Y + Y \otimes \nabla^g_X Y + 2\langle \nabla^g_X Y, Y\rangle \in \Gamma(I(M, g)). \qquad \square$$

Definition 4.1.3. Let (S, γ) be a Dirac bundle. A **Dirac connection** on S is a metric connection ∇ on S such that

$$\nabla_X(a \cdot \psi) = \nabla^g_X a \cdot \psi + a \cdot \nabla_X \psi.$$

A triple (S, γ, ∇) where ∇ is a Dirac connection for the Dirac bundle (S, γ) we call a **Dirac bundle with connection**.

Viewing γ as a section of $\mathrm{Cl}(M,g)^* \otimes \mathrm{End}(S)$, the Dirac connection condition is equivalent to the condition $\nabla\gamma = 0$, where ∇ is the induced connection on $\mathrm{Cl}(M,g)^* \otimes \mathrm{End}(S)$.

Lemma 4.1.4. *If (S,γ) is a graded Dirac bundle and ∇ is a Dirac connection on S, then ∇ preserves the grading.*

Proof. The grading decomposition on S is always given as the eigenbundles of a chirality element in $\mathbb{C}\mathrm{l}(M,g)$, and chirality elements are always parallel, since Riemannian volume forms are Levi-Civita parallel. □

Example 4.1.5.

1. $\mathrm{Cl}(M,g)$ is a real Dirac bundle with connection when endowed with the Levi-Civita connection and the metric $\langle a,b\rangle := \mathrm{tr}(a^t b)$. Similarly, $\mathbb{C}\mathrm{l}(M,g)$ is a complex Dirac bundle with the Hermitian metric $h(a,b) := \mathrm{tr}(a^* b)$.
2. $\Omega(M)$ is a real Dirac bundle with connection when endowed with the Levi-Civita connection and the metric inherited from TM. Of course, $\Omega(M,\mathbb{C})$ is also a complex Dirac bundle.

Definition 4.1.6. Let (S,γ,∇) be a Dirac bundle with connection. We define its **Dirac operator** $\not{D} : \Gamma(S) \to \Gamma(S)$ as the composition

$$\Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\gamma} \Gamma(S).$$

We can express it locally as

$$\not{D}\psi = \sum_i E_i \cdot \nabla_{E_i}\psi,$$

for any local orthonormal frame $(E_i)_i$.

Example 4.1.7. Let $\alpha \in \Omega^k(M)$, then the Dirac operator on α can be computed as

$$\not{D}\alpha = \sum_i (E_i \wedge \nabla_{E_i}\alpha - i_{E_i}\nabla_{E_i}\alpha) = (d + d^*)\alpha.$$

Observe that for a graded Dirac bundle $S = S^+ \oplus S^-$ the Dirac operator breaks into two pieces $\not{D}_\pm : \Gamma(S^\pm) \to \Gamma(S^\mp)$.

For the proof of the following proposition, recall that the divergence of a vector field $X \in \mathfrak{X}(M)$ is defined as

$$\mathrm{div}^g X := \mathrm{tr}(\nabla^g X) = \sum_i \langle \nabla^g_{E_i} X, E_i\rangle \in C^\infty(M,\mathbb{R}).$$

It also satisfies $\mathcal{L}_X \mathrm{vol}^g = (\mathrm{div}^g X)\mathrm{vol}^g$:

$$\begin{aligned}\mathcal{L}_X \mathrm{vol}^g = \mathcal{L}_X(E^1 \wedge \cdots \wedge E^n) &= -\sum_i \langle [X,E_i], E_i\rangle \mathrm{vol}^g \\ &= \sum_i \langle \nabla^g_{E_i} X - \nabla^g_X E_i, E_i\rangle \mathrm{vol}^g = (\mathrm{div}^g X)\mathrm{vol}^g.\end{aligned}$$

Lastly, by Stokes's theorem, if $\omega \in \Omega^n(M)$, then

$$\int_M \mathcal{L}_X\omega = \int_M d i_X\omega = \int_{\partial M} i_X\omega = 0.$$

Proposition 4.1.8. *On a compact manifold, the Dirac operator is formally self-adjoint.*

Proof. Let $\psi, \varphi \in \Gamma(S)$. Then

$$\begin{aligned}\langle \not{D}\psi, \varphi\rangle_2 &= \int_M \langle \not{D}\psi, \varphi\rangle \mathrm{vol}^g = \sum_i \int_M \langle E_i \cdot \nabla_{E_i}\psi, \varphi\rangle \mathrm{vol}^g \\ &= -\sum_i \int_M \langle \nabla_{E_i}\psi, E_i \cdot \varphi\rangle \mathrm{vol}^g \\ &= \langle \psi, \not{D}\varphi\rangle_2 - \sum_i \left(\int_M \mathcal{L}_{E_i} \langle \psi, E_i \cdot \varphi\rangle \mathrm{vol}^g - \int_M \langle \psi, \nabla^g_{E_i} E_i \cdot \varphi\rangle \mathrm{vol}^g \right).\end{aligned}$$

The result now follows from the following computation:

$$\begin{aligned}\sum_i \int_M \mathcal{L}_{E_i} \langle \psi, E_i \cdot \varphi\rangle \mathrm{vol}^g &= \sum_i \left(\int_M \mathcal{L}_{E_i} (\langle \psi, E_i \cdot \varphi\rangle \mathrm{vol}^g) - \int_M \langle \psi, E_i \cdot \varphi\rangle \mathcal{L}_{E_i} \mathrm{vol}^g \right) \\ &= -\sum_i \int_M \langle \psi, E_i \cdot \varphi\rangle (\mathrm{div}^g E_i) \mathrm{vol}^g \\ &= -\sum_{i,j} \int_M \langle \psi, E_i \cdot \varphi\rangle \langle \nabla^g_{E_j} E_i, E_j\rangle \mathrm{vol}^g \\ &= \sum_{i,j} \int_M \langle \psi, E_i \cdot \varphi\rangle \langle E_i, \nabla^g_{E_j} E_j\rangle \mathrm{vol}^g \\ &= \sum_j \int_M \langle \psi, \nabla^g_{E_j} E_j \cdot \varphi\rangle \mathrm{vol}^g . \square\end{aligned}$$

Let us now study the symbol of $\not{D}$.

Definition 4.1.9. Let $(E, \nabla) \to M$ be a vector bundle with connection. Then the operator $\nabla^* : \Omega^1(M, E) \to \Gamma(E)$ is defined as

$$\nabla^* \alpha := -\operatorname{tr}^g(\nabla \alpha), \qquad \text{for } \alpha \in \Omega^1(M, E),$$

where $\operatorname{tr}^g : T^*M \otimes T^*M \otimes E \to E$ is the trace in the first two components.

Explicitly, we can write it as

$$\nabla^* \alpha = -\sum_i \nabla_{E_i} \alpha(E_i) = -\sum_i (\nabla_{E_i}(\alpha(E_i)) - \alpha(\nabla^g_{E_i} E_i)).$$

If E is equipped with a metric with which ∇ is compatible, then ∇^* is actually the formal adjoint of $\nabla : \Gamma(E) \to \Omega^1(M, E)$.

Proposition 4.1.10. *Let (S, γ, ∇) be a Dirac bundle with connection. Then, if $\xi \in T^*M$, $\psi \in S$ and $\alpha \in T^*M \otimes S$,*

$$\begin{aligned}\sigma_1(\not{D})(\xi)\psi &= -i\xi \cdot \psi, \\ \sigma_1(\nabla)(\xi)\psi &= -i\xi \otimes \psi, \\ \sigma_1(\nabla^*)(\xi)\alpha &= i\alpha(\xi).\end{aligned}$$

In particular, $\sigma_2(\not{D}^2)(\xi) = \sigma_2(\nabla^*\nabla)(\xi) = \|\xi\|^2$.

Proof. Let $\xi = df$ for $f \in C^\infty(M, \mathbb{R})$. Then

$$\begin{aligned}
i\sigma_1(\not{D})(\xi)\psi &= \not{D}(f\psi) - f\not{D}\psi = \gamma(df \otimes \psi + f\nabla\psi) - f\gamma(\nabla\psi) = \xi \cdot \psi, \\
i\sigma_1(\nabla)(\xi)\psi &= \nabla(f\psi) - f\nabla\psi = df \otimes \psi = \xi \otimes \psi, \\
i\sigma_1(\nabla^*)(\xi)\alpha &= \nabla^*(f\alpha) - f\nabla^*\alpha = -\operatorname{tr}^g(df \otimes \alpha + f\nabla\alpha) + f\operatorname{tr}^g(\nabla\alpha) \\
&= -\alpha(df) = -\alpha(\xi).
\end{aligned}$$

In particular:

$$\begin{aligned}
\sigma_2(\not{D}^2)(\xi)\psi &= \sigma_1(\not{D})(\xi)(\sigma_1(\not{D})(\xi)\psi) = -\xi^2 \cdot \psi = \|\xi\|^2\psi, \\
\sigma_2(\nabla^*\nabla)(\xi)\psi &= \sigma_1(\nabla^*)(\xi)(\sigma_1(\nabla)(\xi)\psi) = (\xi \otimes \psi)(\xi) = \|\xi\|^2\psi. \square
\end{aligned}$$

Hence, the difference $\mathscr{R} := \not{D}^2 - \nabla^*\nabla$ is a first order differential operator. It is a key fact that it is actually zeroth order, that is, a tensor.

Theorem 4.1.11 (Bochner formula)**.** *Let (S, γ, ∇) be Dirac bundle with connection. Then*

$$\not{D}^2 = \nabla^*\nabla + \mathscr{R},$$

where $\mathscr{R} \in \operatorname{End}(S)$ is given by

$$\mathscr{R}\psi = \frac{1}{2}\sum_{i,j} E_i \cdot E_j \cdot R^\nabla(E_i, E_j)\psi.$$

Proof. First of all, note that $\nabla^*\nabla\psi = -\operatorname{tr}^g(\nabla^2\psi)$, where

$$\nabla^2_{X,Y}\psi := \nabla_X\nabla_Y\psi - \nabla_{\nabla^g_X Y}\psi.$$

Notice as well that $(\nabla^2_{X,Y} - \nabla^2_{Y,X})\psi = R^\nabla(X, Y)\psi$. We can now compute:

$$\begin{aligned}
\not{D}^2\psi &= \sum_{i,j} E_i \cdot \nabla_{E_i}(E_j \cdot \nabla_{E_j}\psi) = \sum_{i,j}\Big(E_i \cdot \nabla^g_{E_i}E_j \cdot \nabla_{E_j}\psi + E_i \cdot E_j \cdot \nabla_{E_i}\nabla_{E_j}\psi\Big) \\
&= \sum_{i,j} E_i \cdot E_j \cdot \nabla^2_{E_i,E_j}\psi + \sum_{i,j}\Big(E_i \cdot \nabla^g_{E_i}E_j \cdot \nabla_{E_j}\psi + E_i \cdot E_j \cdot \nabla_{\nabla^g_{E_i}E_j}\psi\Big) \\
&= -\sum_i \nabla^2_{E_i,E_i}\psi + \frac{1}{2}\sum_{i \neq j} E_i \cdot E_j \cdot (\nabla^2_{E_i,E_j} - \nabla^2_{E_j,E_i})\psi \\
&\qquad + \sum_{i,j}\Big(E_i \cdot \nabla^g_{E_i}E_j \cdot \nabla_{E_j}\psi + E_i \cdot E_j \cdot \nabla_{\nabla^g_{E_i}E_j}\psi\Big) \\
&= \nabla^*\nabla\psi + \frac{1}{2}\sum_{i,j} E_i \cdot E_j \cdot R^\nabla(E_i, E_j)\psi \\
&\qquad + \sum_{i,j}\Big(E_i \cdot \nabla^g_{E_i}E_j \cdot \nabla_{E_j}\psi + E_i \cdot E_j \cdot \nabla_{\nabla^g_{E_i}E_j}\psi\Big).
\end{aligned}$$

The last sum, call it A, is actually zero:

$$\begin{aligned}
A &= \sum_{i,j}\Big(E_i \cdot \nabla^g_{E_i}E_j \cdot \psi + E_i \cdot E_j \cdot \nabla_{\nabla^g_{E_i}E_j}\psi\Big) \\
&= \sum_{i,j,k}\Big(\langle\nabla^g_{E_i}E_j, E_k\rangle E_i \cdot E_k \cdot \nabla_{E_j}\psi + \langle\nabla^g_{E_i}E_j, E_k\rangle E_i \cdot E_j \cdot \nabla_{E_k}\psi\Big) \\
&= \sum_{i,j,k}(\langle\nabla^g_{E_i}E_k, E_j\rangle + \langle\nabla^g_{E_i}E_j, E_k\rangle)E_i \cdot E_j \cdot \nabla_{E_k}\psi = 0. \square
\end{aligned}$$

We would like to particularize now Bochner's formula the special case of spinor bundles.

4.1.2 Spinor bundles

Definition 4.1.12. A complex **spinor bundle** is a complex graded Dirac bundle (S, γ) such that the Clifford action $\gamma : \mathbb{C}\mathrm{l}(M, g) \to \mathrm{End}(S)$ is an isomorphism of algebra bundles. A Dirac connection for a spinor bundle will be called a **spinorial connection**. If S is a spinor bundle, we denote by $\mathscr{A}(S)$ the space of spinorial connections on S.

Spinor bundles here will always be complex, so we will drop the adjective complex.

Lemma 4.1.13. *If S is a spinor bundle, then $\mathscr{A}(S)$ is an affine space modeled on $i\Omega^1(M)$.*

Proof. Let $\nabla' = \nabla + B$, with $\nabla', \nabla \in \mathscr{A}(S)$ and $B \in \Omega^1(M, \mathrm{End}(S))$. Then for all $X \in \mathfrak{X}(M)$ and $a \in \Gamma(\mathbb{C}\mathrm{l}(M, g))$, since both ∇ and ∇' are compatible with γ:

$$\nabla_X \gamma_a = \gamma_{\nabla^g_X a} = \nabla'_X \gamma_a = \nabla_X \gamma_a + [B_X, \gamma_a],$$

so B_X lies in the center of $\mathrm{End}(S)$ at each point, which means that actually $B \in \Omega^1(M, \mathbb{C})$. Moreover, since both ∇' and ∇ are compatible with the Hermitian metric, we get that $0 = B + B^* = B + \overline{B}$, which means that $B \in i\Omega^1(M)$. □

We now want to see that $\mathscr{A}(S)$ is actually non-empty as well.

Lemma 4.1.14. *Let $S \to M$ be a complex vector bundle, then every algebra derivation of $\mathrm{End}(S)$ is inner. More concretely, if $\delta \in \mathrm{End}(\mathrm{End}(S))$ is such that $\delta(AB) = (\delta A)B + A\delta B$, then there is $C \in \mathrm{End}(S)$ such that $\delta = [C, \cdot]$. Moreover, if S is endowed with a Hermitian metric and δ is such that $\delta A^* = (\delta A)^*$, then there is $C \in \mathfrak{u}(S)$ such that $\delta = [C, \cdot]$.*

Proof. Let $\{\psi_i\}_i$ and $\{\lambda^i\}_i$ be collections in $\Gamma(S)$ and $\Gamma(S^*)$, respectively, such that $\sum_i \lambda^i(\psi_i) = 1$ (using, say, local frames and partitions of unity), and define

$$C\varphi := \sum_i \delta(\varphi \otimes \lambda^i)\psi_i.$$

Then

$$\begin{aligned}
[C, A]\varphi &= CA\varphi - AC\varphi \\
&= \sum_i \Big(\delta(A\varphi \otimes \lambda^i)\psi_i - A\delta(\varphi \otimes \lambda^i)\psi_i\Big) \\
&= \sum_i \Big(\delta(A \circ (\varphi \otimes \lambda^i))\psi_i - A\delta(\varphi \otimes \lambda^i)\psi_i\Big) \\
&= \sum_i (\delta A)(\lambda^i(\psi_i)\varphi) = (\delta A)\varphi.
\end{aligned}$$

If S comes with a Hermitian metric and δ is such that $\delta A^* = (\delta A)^*$, then we have that

$$0 = \delta A^* - (\delta A)^* = [C, A^*] - [C, A]^* = [C + C^*, A^*],$$

for all $A \in \mathrm{End}(S)$. Hence, $C + C^*$ lies in the center of $\mathrm{End}(S)$, so there is $\lambda \in C^\infty(M)$ such that $C + C^* = \lambda$. Since $C + C^*$ is self-adjoint, we actually have that $\lambda = \overline{\lambda}$, so that $\lambda \in C^\infty(M, \mathbb{R})$. Consider now $\hat{C} := C - \frac{\lambda}{2}$. Then $\delta = [C, \cdot] = [\hat{C}, \cdot]$, and $\hat{C} + \hat{C}^* = C + C^* - \lambda = 0$. □

Lemma 4.1.15. *Let $S \to M$ be a complex vector bundle, then every connection on* $\mathrm{End}(S)$ *compatible with the algebra structure, in the sense that ∇_X is an algebra derivation of* $\mathrm{End}(S)$ *for all $X \in \mathfrak{X}(M)$, comes from an connection on S. If moreover S is equipped with a Hermitian metric and ∇ is unitary, in the sense that $(\nabla_X A)^* = \nabla_X(A^*)$ for all $A \in \mathrm{End}(S)$, then it comes from a unitary connection on S.*

Proof. Let ∇ be a connection on $\mathrm{End}(S)$ compatible with the algebra structure, and let ∇^0 be any connection on S. Denote by $\hat{\nabla}^0$ the connection on $\mathrm{End}(S)$ induced by ∇^0, which is also compatible with the algebra structure. Then $\nabla = \hat{\nabla}^0 + B$, for $B \in \Omega^1(M, \mathrm{End}(\mathrm{End}(S)))$, and since both ∇ and $\hat{\nabla}^0$ are compatible with the algebra structure, we have that

$$B_X(AC) = (B_X A)C + AB_X C, \qquad \text{for } X \in TM \text{ and } A, C \in \mathrm{End}(S).$$

Hence, B actually takes values in the algebra derivations of $\mathrm{End}(S)$. By Lemma 4.1.14, then, there is $\tilde{B} \in \Omega^1(M, \mathrm{End}(S))$ such that $B_X = [\tilde{B}_X, \cdot]$. Consider $\tilde{\nabla} := \nabla + \tilde{B}$, which induces ∇ on $\mathrm{End}(S)$.

If S comes with a Hermitian metric and ∇ is unitary, we can run the previous argument with ∇^0 a unitary connection on S. This gives that

$$(B_X A)^* = B_X A^*, \qquad \text{for } X \in TM \text{ and } A \in \mathrm{End}(S),$$

so that by Lemma 4.1.14 again we have that $B_X = [\tilde{B}_X, \cdot]$ for $\tilde{B} \in \Omega^1(M, \mathrm{End}(S))$ such that $B_X + B_X^* = 0$. Then $\tilde{\nabla} := \nabla + \tilde{B}$ is a Hermitian metric on S inducing ∇ on $\mathrm{End}(S)$. □

Proposition 4.1.16. *Let S be a spinor bundle. Then $\mathscr{A}(S)$ is a non-empty affine space modeled on $i\Omega^1(M)$. Moreover, if $\nabla', \nabla \in \mathscr{A}(S)$ with $\nabla' = \nabla + i\alpha$, for $\alpha \in \Omega^1(M)$, then*

$$\not{D}'\psi = \not{D}\psi + i\alpha \cdot \psi, \qquad \textit{for } \psi \in \Gamma(S).$$

Proof. Let ∇^0 be any unitary connection on S. Define $\delta \in \Omega^1(M, \mathrm{End}(\mathrm{End}(S)))$ by

$$\delta_X \gamma_a := \gamma_{\nabla^g_X a} - \nabla^0_X \gamma_a, \qquad \text{for } X \in TM.$$

One can easily check that δ_X is actually an algebra derivation of $\mathrm{End}(S)$. Moreover, it preserves adjoints: if $a \in \mathbb{C}\mathrm{l}(M, g)$ then

$$\begin{aligned}\delta_X \gamma_a^* = \delta_X \gamma_{\epsilon(a^*)} &= \gamma_{\nabla^g_X(\epsilon(a^*))} - \nabla^0_X \gamma_{\epsilon(a^*)} \\ &= \gamma_{\epsilon((\nabla^g_X a)^*)} - \nabla^0_X \gamma_a^* \\ &= (\delta_X \gamma_a)^*.\end{aligned}$$

By Lemma 4.1.14, there is $B \in \Omega^1(M, \mathrm{End}(S))$ such that $\delta_X = [B_X, \cdot]$ and $B_X + B_X^* = 0$. Consider now $\nabla := \nabla^0 + B$. It is clearly Hermitian. It is also compatible with the Clifford action: if $a \in \mathbb{C}\mathrm{l}(M, g)$, then

$$\nabla_X \gamma_a = \nabla^0_X \gamma_a + [B_X, \gamma_a] = \nabla^0_X \gamma_a + \delta_X \gamma_a = \gamma_{\nabla^g_X a}.$$

Hence, $\nabla \in \mathscr{A}(S)$.

If $\nabla' = \nabla + i\alpha$, with $\nabla', \nabla \in \mathscr{A}(S)$ and $\alpha \in \Omega^1(M)$, then

$$\not{D}'\psi = \gamma(\nabla'\psi) = \gamma(\nabla\psi + i\alpha \otimes \psi) = \not{D}\psi + i\alpha \cdot \psi. \qquad \square$$

In what follows we will identify $\mathbb{C}\mathrm{l}(M,g)$ with $\Omega(M,\mathbb{C})$ via the symbol and quantization maps and the metric without further mention. By this we mean that if $\omega \in \Omega^2(M,\mathbb{C})$ and $\psi \in \Gamma(S)$, then by $\omega \cdot \psi$ we mean $q(\omega) \cdot \psi$, where q is the quantization map, and where ω is considered as an element of $\mathfrak{X}^2(M,\mathbb{C})$ using the metric g. We will also make the identifications $\wedge^2 T^*M \cong \wedge^2 TM \cong \mathfrak{so}(TM)$, via g. From this perspective, if $X, Y \in TM$, then $X \wedge Y \in \mathfrak{so}(TM)$ is given by

$$(X \wedge Y)Z = \langle X, Z\rangle Y - \langle Y, Z\rangle X.$$

Notice that if $A \in \mathfrak{so}(TM)$, then we can extend A to $\mathrm{Cl}(M,g)$, since A preserves the ideal $I(M,g)$:

$$A(X \otimes X + \|X\|^2) = AX \otimes X + X \otimes AX + 2\langle AX, X\rangle \in I(M,g),$$

since $\langle AX, X\rangle = 0$.

Lemma 4.1.17. *If $X, Y \in TM$, then $(X \wedge Y)a = \frac{1}{2}[X \wedge Y, a]$, for all $a \in \mathrm{Cl}(M,g)$.*

Proof. If $Z \in TM$, then

$$\begin{aligned}(X \wedge Y)Z &= \langle X, Z\rangle Y - \langle Y, Z\rangle X \\ &= \frac{1}{2}(-XZY - ZXY + XYZ + XZY) \\ &= \frac{1}{2}[XY, Z] = \frac{1}{2}[X \wedge Y, Z].\end{aligned}$$

If now $Z_i \in TM$, then

$$\begin{aligned}(X \wedge Y)Z_1 \dots Z_k &= \sum_i Z_1 \dots Z_{i-1} \cdot (X \wedge Y)Z_i \cdot Z_{i+1} \dots Z_k \\ &= \frac{1}{2}\sum_i Z_1 \dots Z_{i-1}[XY, Z_i]Z_{i+1} \dots Z_k \\ &= \frac{1}{2}(XYZ_1 \dots Z_k - Z_1 \dots Z_k XY) \\ &= \frac{1}{2}[XY, Z_1 \dots Z_k] = \frac{1}{2}[X \wedge Y, Z_1 \dots Z_k].\end{aligned}$$

□

Lemma 4.1.18. *If $(S = S^+ \oplus S^-, \gamma)$ is a graded Dirac bundle, then $\gamma : \mathbb{C}\mathrm{l}(M,g) \to \mathrm{End}(S)$ preserves traces, in the sense that $\mathrm{tr}(\gamma_a) = s\,\mathrm{tr}(a)$, for $a \in \mathbb{C}\mathrm{l}(M,g)$ and where $s = \mathrm{rk}_{\mathbb{C}} S$. In particular, γ is an isometry, where the Hermitian product on $\mathrm{End}(S)$ is given by $(A, B) = \frac{1}{s}\mathrm{tr}(A^*B)$.*

In particular, if $\omega \in \Omega^k(M,\mathbb{C})$ and $\eta \in \Omega^\ell(M,\mathbb{C})$, then

$$\mathrm{tr}(\gamma_\omega^* \gamma_\eta) = \begin{cases} 0, & k \neq \ell, \\ s\langle \overline{\omega}, \eta\rangle, & k = \ell. \end{cases}$$

Proof. First of all, if $a = 1$, then $\mathrm{tr}(\gamma_1) = s = s\,\mathrm{tr}(1)$. If $(e_i)_i$ is an orthonormal basis for T_xM, then if k is odd, $\gamma_{e_{i_1} \dots e_{i_k}} : S^\pm \to S^\mp$, so $\mathrm{tr}(\gamma_{e_{i_1} \dots e_{i_k}}) = 0$. On the other hand, if k is even, then

$$\mathrm{tr}(\gamma_{e_{i_1} \dots e_{i_k}}) = \mathrm{tr}(\gamma_{e_{i_2} \dots e_{i_k} e_{i_1}}) = -\mathrm{tr}(\gamma_{e_{i_1} \dots e_{i_k}}),$$

so $\operatorname{tr}(\gamma_{e_{i_1}\dots e_{i_k}}) = 0$ as well. On the other hand,

$$\operatorname{tr}(e_{i_1}\dots e_{i_k}) = \sigma(e_{i_1}\dots e_{i_k})_0 = (e_{i_1}\wedge\cdots\wedge e_{i_k})_0 = 0.$$

Hence, γ indeed preserves traces. This directly gives that γ is an isometry.

In particular, if $\omega \in \Omega^k(M,\mathbb{C})$ and $\eta \in \Omega^\ell(M,\mathbb{C})$, then

$$\operatorname{tr}(\gamma_\omega^*\gamma_\eta) = s\operatorname{tr}(\overline{\omega}^t\eta) = \begin{cases} 0, & k \neq \ell, \\ s\langle\overline{\omega},\eta\rangle, & k = \ell. \end{cases}$$ □

Proposition 4.1.19. *If S is a spinor bundle and $\nabla \in \mathscr{A}(S)$, then there is a purely imaginary 2-form $F^\nabla \in i\Omega^2(M)$ such that*

$$R^\nabla(X,Y) = \frac{1}{2}\gamma_{R^g(X,Y)} + \frac{2}{s}F^\nabla(X,Y), \qquad \text{for } X,Y \in TM,$$

where $s := \operatorname{rk}_{\mathbb{C}} S$. Explicitly, $F^\nabla = \frac{1}{2}\operatorname{tr}(R^\nabla)$. If moreover $S = S^+\oplus S^-$ is graded, then $F^\nabla = \operatorname{tr}(R^{\nabla^+})$ as well, where ∇^+ is the restriction of ∇ to S^+.

Proof. If $a \in \mathrm{Cl}(M,g)$, then

$$\begin{aligned}
[R^\nabla(X,Y),\gamma_a] &= [[\nabla_X,\nabla_Y] - \nabla_{[X,Y]},\gamma_a] \\
&= -[[\nabla_Y,\gamma_a],\nabla_X] - [[\gamma_a,\nabla_X],\nabla_Y] - \gamma_{\nabla^g_{[X,Y]}a} \\
&= -[\gamma_{\nabla^g_Y a},\nabla_X] + [\gamma_{\nabla^g_X a},\nabla_Y] - \gamma_{\nabla^g_{[X,Y]}a} \\
&= \gamma_{R^g(X,Y)a} = \frac{1}{2}[\gamma_{R^g(X,Y)},\gamma_a].
\end{aligned}$$

Since S is a spinor bundle, this means that $R^\nabla(X,Y) - \frac{1}{2}\gamma_{R^g(X,Y)}$ is in the center of $\operatorname{End}(S)$. Hence, there is a two form $F^\nabla \in \Omega^2(M,\mathbb{C})$ such that

$$R^\nabla(X,Y) = \frac{1}{2}\gamma_{R^g(X,Y)} + \frac{2}{s}F^\nabla(X,Y).$$

By Lemma 4.1.18, taking the trace of this equality gives that $2F^\nabla = \operatorname{tr}(R^\nabla)$, which is purely imaginary, since ∇ is unitary.

Assume now that $S = S^+\oplus S^-$. If $Z \in TM$, then the previous computation gives that, restricting to S^+:

$$[R^\nabla(X,Y),\gamma_Z] = R^{\nabla^-}(X,Y)\gamma_Z - \gamma_Z R^{\nabla^+}(X,Y) = \gamma_{R^g(X,Y)Z},$$

so that, again by Lemma 4.1.18,

$$\begin{aligned}
0 = -s\langle R^g(X,Y)Z,Z\rangle &= \operatorname{tr}(\gamma_{Z\cdot R^g(X,Y)Z}) \\
&= \operatorname{tr}(\gamma_Z R^{\nabla^-}(X,Y)\gamma_Z + \|Z\|^2 R^{\nabla^+}(X,Y)) \\
&= \|Z\|^2\left(-\operatorname{tr}(R^{\nabla^-}(X,Y)) + \operatorname{tr}(R^{\nabla^+}(X,Y))\right).
\end{aligned}$$

Hence, $\operatorname{tr}(R^\nabla) = \operatorname{tr}(R^{\nabla^+}) + \operatorname{tr}(R^{\nabla^-}) = 2\operatorname{tr}(R^{\nabla^+})$. □

Theorem 4.1.20 (Lichnerowicz formula)**.** *If S is a spinor bundle and $\nabla \in \mathscr{A}(S)$, then*

$$D\!\!\!/^2 = \nabla^*\nabla + \frac{1}{4}\mathrm{scal}^g + \frac{2}{s}\gamma_{F^\nabla}.$$

Proof. We need only compute $\mathscr{R}$ from the Bochner formula (Theorem 4.1.11) in this case. Using Proposition 4.1.19 and the algebraic Bianchi identity for R^g:

$$\begin{aligned}
\mathscr{R}\psi &= \frac{1}{2}\sum_{j,k} E_j \cdot E_k \cdot R^{\nabla}(E_j, E_k)\psi \\
&= \frac{1}{2}\sum_{j,k} E_j \cdot E_k \cdot \left(\frac{1}{2}R^g(E_j, E_k)\cdot\psi + \frac{2}{s}F^{\nabla}(E_j, E_k)\psi\right) \\
&= \frac{1}{8}\sum_{j,k,l,m} \langle R^g(E_j, E_k)E_l, E_m\rangle E_j E_k E_l E_m \cdot \psi + \frac{2}{s}F^{\nabla}\cdot\psi \\
&= \frac{1}{24}\sum_{j\neq k\neq l\neq j,m} (\langle R^g(E_j, E_k)E_l, E_m\rangle + \langle R^g(E_k, E_l)E_j, E_m\rangle \\
&\qquad + \langle R^g(E_l, E_j)E_k, E_m\rangle) E_j E_k E_l E_m \cdot \psi \\
&\qquad + \frac{1}{8}\sum_{j,k,m} \langle R^g(E_j, E_k)E_j, E_m\rangle E_k E_m \cdot \psi \\
&\qquad - \frac{1}{8}\sum_{j,k,m} \langle R^g(E_j, E_k)E_k, E_m\rangle E_j E_m \cdot \psi + \frac{2}{s}F^{\nabla}\cdot\psi \\
&= -\frac{1}{4}\sum_{j,m} \mathrm{Ric}^g(E_j, E_m)E_j E_m \cdot \psi + \frac{2}{s}F^{\nabla}\cdot\psi \\
&= \frac{1}{4}\mathrm{scal}^g\psi + \frac{2}{s}F^{\nabla}\cdot\psi.
\end{aligned}$$

□

4.2 The Seiberg–Witten equations

In this whole section $(S = S^+ + S^-, \gamma)$ will always be a graded spinor bundle and M will be connected, orientable and closed.

4.2.1 The equations

For every $\psi \in \Gamma(S)$, we can consider the endomorphism $\overline{\psi}\otimes\psi \in \mathrm{End}(S)$, by using the Hermitian metric on S, by which we mean

$$(\overline{\psi}\otimes\psi)\varphi := \langle\psi, \varphi\rangle\psi.$$

It is of course self-adjoint. Its trace is given by:

$$\mathrm{tr}(\overline{\psi}\otimes\psi) = \sum_i |\langle\psi, \varphi_i\rangle|^2 = \|\psi\|^2,$$

where $(\varphi_i)_i$ is any local orthonormal frame for S. Hence, $\overline{\psi}\otimes\psi - \frac{1}{s}\|\psi\|^2$ is a self-adjoint traceless endomorphism of S, where $s := \mathrm{rk}_{\mathbb{C}}\, S$.

From now on, M will be 4-dimensional, in which case $S^{\pm}$ become rank 2. In such case, if $\psi \in \Gamma(S^+)$, then $\overline{\psi}\otimes\psi - \frac{1}{2}\|\psi\|^2$ is a self-adjoint traceless endomorphism of S^+. Recall that in 4-dimensions we have

$$\mathrm{End}(S^+) \cong \mathrm{Cl}^0_+(M, g) \cong C^\infty(M)(1 + \Gamma_c) \oplus \Omega^2_+(M, \mathbb{C}).$$

Under such identifications, $\mathrm{End}_0(S^+) \cong \Omega^2_+(M,\mathbb{C})$ (cf. Lemma 4.1.18), where $\mathrm{End}_0(S^+)$ are traceless endormophisms. Hence, we can think of $\overline{\psi} \otimes \psi - \frac{1}{2}\|\psi\|^2$ as a self-dual complex 2-form on M. Moreover, it is purely imaginary. Indeed, if $\omega \in \Omega^2(M,\mathbb{C})$ is such that γ_ω is self-adjoint, then

$$\gamma_\omega = \gamma_\omega^* = -\gamma_{\overline{\omega}},$$

so $\omega = -\overline{\omega}$.

Definition 4.2.1. The (perturbed) **Seiberg–Witten equations** for a pair $(\nabla, \psi) \in \mathscr{A}(S) \times \Gamma(S^+)$ and perturbation parameter $\eta \in \Omega^2(M)$, with $d\eta = 0$, are

$$F_+^\nabla = \overline{\psi} \otimes \psi - \frac{1}{2}\|\psi\|^2 - i\eta_+, \tag{4.1}$$

$$\not{D}\psi = 0. \tag{4.2}$$

We define the **configuration space** $\mathscr{C}(S) := \mathscr{A}(S) \times \Gamma(S^+)$, the **target space** $\mathscr{Y}(S) := i\Omega^2_+(M) \times \Gamma(S^-)$, the **Seiberg–Witten map** $\mathrm{SW}_\eta : \mathscr{C}(S) \to \mathscr{Y}(S)$ given by $\mathrm{SW}_\eta(\nabla,\psi) := (F_+^\nabla - q(\psi) + i\eta, \not{D}\psi)$, for $q(\psi) := \overline{\psi} \otimes \psi - \frac{1}{2}\|\psi\|^2$, the **solution space** $\mathscr{Z}_\eta(S) := \mathrm{SW}_\eta^{-1}(0,0) = \mathrm{SW}_0^{-1}(-i\eta_+, 0)$, and the **gauge group** $\mathscr{G}(M) := C^\infty(M, \mathrm{U}(1))$.

Explicitly, (4.1) means that for all $\varphi \in S$,

$$F_+^\nabla \cdot \varphi = \langle \psi, \varphi \rangle \psi - \frac{1}{2}\|\psi\|^2\varphi - i\eta_+ \cdot \varphi.$$

We now begin the study of the moduli space of solutions to the Seiberg–Witten equations.

Lemma 4.2.2. *The maps*

$$\begin{array}{ccc} \mathscr{G}(M) \times \mathscr{C}(S) & \longrightarrow & \mathscr{C}(S) \\ (u, \nabla, \psi) & \longmapsto & (\nabla - u^{-1}du, u\psi) \end{array}$$

and

$$\begin{array}{ccc} \mathscr{G}(M) \times \mathscr{Y}(S) & \longrightarrow & \mathscr{Y}(S) \\ (u, \omega, \varphi) & \longmapsto & (\omega, u\varphi) \end{array}$$

define left actions with respect to which SW_η *is equivariant.*

Moreover, if $(\nabla,\psi) \in \mathscr{C}(S)$*, then its stabilizer* $\mathrm{Stab}(\nabla,\psi) \subseteq \mathscr{G}(M)$ *is either trivial, if* $\psi \neq 0$*, or* $\mathrm{U}(1)$*, if* $\psi = 0$.

Proof. That the action on $\mathscr{C}(S)$ is indeed a left action follows easily from

$$(uv)^{-1}d(uv) = v^{-1}u^{-1}(vdu + udv) = u^{-1}du + v^{-1}dv,$$

for $u, v \in \mathscr{G}(M)$. To see that SW_η is equivariant, first notice that $\nabla - u^{-1}du = u\nabla u^{-1}$, so $R^{\nabla - u^{-1}du} = uR^\nabla u^{-1}$, and hence $F^{\nabla - u^{-1}du} = F^\nabla$. On the other hand, $q(u\psi) = q(\psi)$. Secondly, by Proposition 4.1.16, if $\nabla' = \nabla - u^{-1}du$, then

$$\not{D}'(u\psi) = \not{D}(u\psi) - u^{-1}du \cdot (u\psi) = u\not{D}\psi.$$

Let now $u \in \mathrm{Stab}(\nabla,\psi)$. Then $\nabla - u^{-1}du = \nabla$ if and only if u is constant, i.e., $u \in \mathrm{U}(1)$. But then $u\psi = \psi$ implies that u can only be non-trivial if $\psi = 0$. □

Definition 4.2.3. We define the Seiberg–Witten **moduli space** as $\mathscr{M}_\eta(S) := \mathscr{Z}_\eta(S)/\mathscr{G}(M)$.

A solution (∇,ψ) is called **reducible** if $\mathrm{Stab}(\nabla,\psi) = \mathrm{U}(1)$ (equivalently, if $\psi = 0$) and **irreducible** if $\mathrm{Stab}(\nabla,\psi) = 1$ (equivalently, if $\psi \neq 0$).

4.2.2 Seiberg–Witten functional

Although the Seiberg–Witten equations cannot be the Euler–Lagrange equations for any functional, since they are first-order equations, they actually do arise from a variational setting.

Definition 4.2.4. We define the **Seiberg–Witten functional** $\mathscr{S} : \mathscr{C}(S) \to \mathbb{R}$ by

$$\mathscr{S}(\nabla, \psi) := \int_M \left(\|\nabla\psi\|^2 + \|F_+^\nabla\|^2 + \frac{\mathrm{scal}^g}{4}\|\psi\|^2 + \frac{1}{8}\|\psi\|^4 \right) \mathrm{vol}^g.$$

Lemma 4.2.5. *The Euler–Lagrange equations for the Seiberg–Witten functional are*

$$\begin{aligned} \nabla^*\nabla\psi &= -\frac{1}{4}\left(\mathrm{scal}^g + \|\psi\|^2\right)\psi, \\ d^*F_+^\nabla &= -\frac{1}{2}\mathrm{Re}\langle\nabla\psi, \psi\rangle. \end{aligned}$$

Proof. Let $(\nabla, \psi) \in \mathscr{C}(S)$. If $\varphi \in \Gamma(S)$, then

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathscr{S}(\nabla, \psi + t\varphi) &= \int_M \left(2\langle\nabla\varphi, \nabla\psi\rangle + \frac{\mathrm{scal}^g}{2}\langle\psi, \varphi\rangle + \frac{1}{2}\langle\psi, \varphi\rangle\|\psi\|^2 \right) \mathrm{vol}^g \\ &= 2\int_M \langle\nabla^*\nabla\psi + \frac{1}{4}(\mathrm{scal}^g + \|\psi\|^2)\psi, \varphi\rangle \mathrm{vol}^g, \end{aligned}$$

and if $\alpha \in \Omega^1(M)$, then, using that $F^{\nabla + it\alpha} = F^\nabla + i2td\alpha$,

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathscr{S}(\nabla + it\alpha, \psi) &= \int_M \left(2\,\mathrm{Re}\langle\nabla\psi, i\alpha\otimes\psi\rangle + 4\langle F_+^\nabla, id\alpha\rangle \right) \mathrm{vol}^g \\ &= 4\int_M \langle d^*F_+^\nabla + \frac{1}{2}\mathrm{Re}\langle\nabla\psi, \psi\rangle, i\alpha\rangle \mathrm{vol}^g. \end{aligned}$$

□

Proposition 4.2.6. *The Seiberg–Witten functional can be expressed as*

$$\mathscr{S}(\nabla, \psi) = \int_M \left(\|\not{D}\psi\|^2 + \|F_+^\nabla - q(\psi)\|^2 \right) \mathrm{vol}^g. \tag{4.3}$$

Hence, the lowest possible value of $\mathscr{S}$ is obtained precisely on $\mathscr{Z}(S)$.

Proof. Let $\hat{\mathscr{S}}(\nabla, \psi)$ be the right-hand side of (4.3). First of all, note that

$$q(\psi)^2 = \left(\overline{\psi} \otimes \psi - \frac{1}{2}\|\psi\|^2 \right)^2 = \frac{1}{4}\|\psi\|^4.$$

Then, using Lemma 4.1.18, we find that

$$\begin{aligned} \|q(\psi)\|^2 &= \langle\overline{q(\psi)}, q(\psi)\rangle = \frac{1}{4}\mathrm{tr}(q(\psi)^2) = \frac{1}{8}\|\psi\|^4, \\ \langle F_+^\nabla, q(\psi)\rangle &= \langle\overline{F^\nabla}, q(\psi)\rangle = \frac{1}{4}\mathrm{tr}(\gamma_{F^\nabla} q(\psi)) \\ &= \frac{1}{4}\sum_i \langle\varphi_i, F^\nabla \cdot (\langle\psi, \varphi_i\rangle\psi - \frac{1}{2}\|\psi\|^2\varphi_i)\rangle \\ &= \frac{1}{4}\langle F^\nabla \cdot \psi, \psi\rangle - \frac{1}{8}\|\psi\|^2\,\mathrm{tr}(\gamma_{F^\nabla}) \end{aligned}$$

$$= \frac{1}{4}\langle F^\nabla \cdot \psi, \psi \rangle.$$

Now, using that $\not{D}$ is formally self-adjoint and Lichnerowicz's formula (Theorem 4.1.20), we finally see that

$$\begin{aligned}
\hat{\mathcal{S}}(\nabla, \psi) &= \int_M \left(\langle \not{D}^2\psi, \psi \rangle + \|F_+^\nabla\|^2 + \|q(\psi)\|^2 - 2\langle F_+^\nabla, q(\psi)\rangle \right) \mathrm{vol}^g \\
&= \int_M \Big(\|\nabla\psi\|^2 + \frac{\mathrm{scal}^g}{4}\|\psi\|^2 + \frac{1}{2}\langle F^\nabla \cdot \psi, \psi\rangle + \|F_+^\nabla\|^2 \\
&\qquad + \frac{1}{8}\|\psi\|^4 - \frac{1}{2}\langle F^\nabla \cdot \psi, \psi \rangle \Big) \mathrm{vol}^g \\
&= \mathcal{S}(\nabla, \psi).
\end{aligned}$$

□

4.2.3 The functional set-up

Let $\hat{\nabla} \in \mathscr{A}(S)$ be a spinorial connection, which we will take as a reference point in $\mathscr{A}(S)$. We will denote by Γ_0 and C_0^∞ compactly supported sections and functions.

Definition 4.2.7. Let $1 \leq p \leq \infty$. An **L^p-section** of S is a measurable map $\psi : M \to S$ such that $\psi(x) \in S_x$ for almost all $x \in M$ and $\|\psi\| \in L^p(M, \mathbb{R})$. We denote the space of L^p-sections up to equality almost everywhere by $L^p(S)$. For $\psi \in L^p(S)$, we define its **L^p-norm** by

$$\|\psi\|_p := \left(\int_M \|\psi\|^p \mathrm{vol}^g \right)^{1/p}$$

if $p < \infty$, and $\|\psi\|_\infty := \operatorname{ess\,sup} \|\psi\|$ if $p = \infty$.

If $\psi \in L^1(S)$, we say that ψ is **k-times weakly differentiable** if there is $\alpha \in L^1(T^*M^{\otimes k} \otimes S)$ such that

$$\int_M \langle \psi, (\hat{\nabla}^k)^*\beta \rangle \mathrm{vol}^g = \int_M \langle \alpha, \beta \rangle \mathrm{vol}^g, \qquad \text{for all } \beta \in \Gamma_0(T^*M^{\otimes k} \otimes S),$$

where $\hat{\nabla}^k : \Gamma(S) \to \Gamma(T^*M^{\otimes k} \otimes S)$ is the composition

$$\Gamma(S) \xrightarrow{\hat{\nabla}} \Gamma(T^*M \otimes S) \xrightarrow{\hat{\nabla}} \dots \xrightarrow{\hat{\nabla}} \Gamma(T^*M^{\otimes k} \otimes S).$$

In such case we say that α is the **weak kth derivative** of ψ, and we write $\hat{\nabla}^k\psi := \alpha$.

We define the **Sobolev space** $L^{k,p}(S)$ as the space of sections $\psi \in L^p(S)$ such that ψ is j-times weakly differentiable and $\hat{\nabla}^j\psi \in L^p(T^*M^{\otimes j} \otimes S)$ for all $1 \leq j \leq k$. If $\psi \in L^{k,p}(S)$, we define its **Sobolev norm** by

$$\|\psi\|_{k,p} := \sum_{j=0}^k \|\hat{\nabla}^j\psi\|_p.$$

Proposition 4.2.8. *For all $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, the Sobolev spaces $(L^{k,p}(S), \|\cdot\|_{k,p})$ are Banach spaces.*

Theorem 4.2.9 (Sobolev embedding, Rellich–Kondrachov and Morrey)**.** *1. If $1 \leq p < \infty$, then $\Gamma(S)$ is dense in $L^{k,p}(S)$.*

2. *If $k, \ell \in \mathbb{N}$ are such that $k \geq \ell$ and $1 \leq p, q < \infty$ are such that*

$$\frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{\ell}{n},$$

then $L^{k,p}(S)$ embeds continuously into $L^{\ell,q}(S)$, i.e., $L^{k,p}(S) \subseteq L^{\ell,q}(S)$ and there is a constant $C > 0$ such that

$$\|\psi\|_{k,p} \leq C\|\psi\|_{\ell,q}, \qquad \text{for all } \psi \in L^{k,p}(S).$$

If moreover

$$\frac{1}{p} - \frac{k}{n} < \frac{1}{q} - \frac{\ell}{n},$$

then the embedding $L^{k,p}(S) \subseteq L^{\ell,q}(S)$ is compact, i.e., any bounded sequence in $L^{k,p}(S)$ admits a subsequence convergent in $L^{\ell,q}(S)$.

3. *If $k, \ell \in \mathbb{N}$ and $1 \leq p < \infty$ are such that*

$$\frac{1}{p} - \frac{k}{n} < -\frac{\ell}{n},$$

then $L^{k,p}(S)$ embeds continuously and compactly into $\Gamma^\ell(S)$, the space of C^ℓ-sections of S.

We will use the Sobolev spaces to topologize the moduli space $\mathscr{M}_\eta(S)$.

Definition 4.2.10. We make the following definitions:

$$\begin{aligned}
\mathscr{A}^{k,p}(S) &:= \{\hat{\nabla} + i\alpha : \alpha \in L^{k,p}(T^*M)\},\\
\mathscr{C}^{k,p}(S) &:= \mathscr{A}^{k,p}(S) \times L^{k,p}(S^+),\\
\mathscr{Y}^{k,p}(S) &:= L^{k,p}(i \wedge^2_+ T^*M) \oplus L^{k,p}(S^-),\\
\mathscr{G}^{k,p}(M) &:= \{u \in L^{k,p}(M, \mathbb{C}) : |u(x)| = 1, \text{ for all } x \in M\},
\end{aligned}$$

and for any $\mathscr{N} \in \{\mathscr{A}, \mathscr{C}, \mathscr{Y}, \mathscr{G}\}$, we let $\mathscr{N}^k := \mathscr{N}^{k,2}$.

Lemma 4.2.11. *For every $k \geq 1$, the map $q : \Gamma(S^+) \to i\Omega^2_+(M)$ extends to a smooth map $q : L^{k+1,2}(S^+) \to L^{k,2}(i \wedge^2_+ T^*M)$.*

Proof. Let $\psi \in L^{k+1,2}(S^+)$. We begin by proving that $q(\psi) \in L^{k,2}(i \wedge^2_+ T^*M)$. Firstly,

$$\int_M \|q(\psi)\|^2 \mathrm{vol}^g = \frac{1}{8} \int_M \|\psi\|^4 \mathrm{vol}^g < \infty,$$

since $L^{k+1,2}(S^+) \subseteq L^4(S^+)$ by the Sobolev embedding, because

$$\frac{1}{2} - \frac{k+1}{4} \leq 0 < \frac{1}{4}.$$

Secondly, since $q(\psi) = \overline{\psi} \otimes \psi - \frac{1}{2}\langle \psi, \psi \rangle$, we see that

$$\hat{\nabla}^j(q(\psi)) = \sum_{\ell=0}^{j} \binom{j}{\ell} \left(\hat{\nabla}^\ell \overline{\psi} \otimes \hat{\nabla}^{j-\ell}\psi - \frac{1}{2}\langle \hat{\nabla}^\ell \psi, \hat{\nabla}^{j-\ell}\psi \rangle \right).$$

Hence, there is a constant $C > 0$ such that

$$\|\hat{\nabla}^j(q(\psi))\|^2 \leq C \sum_{\ell,m=0}^{j} \|\hat{\nabla}^\ell \psi\| \|\hat{\nabla}^{j-\ell}\psi\| \|\hat{\nabla}^m \psi\| \|\hat{\nabla}^{j-m}\psi\|.$$

Observe that for all $\ell = 0, \dots, k$ we have that

$$\frac{1}{2} - \frac{k+1-\ell}{4} \leq \frac{1}{4},$$

so $\hat{\nabla}^\ell \psi \in L^{k+1-\ell,2}(S^+) \subseteq L^4(S^+)$. The Hölder inequality now gives that

$$\|\hat{\nabla}^j(q(\psi))\|_2^2 \leq C \sum_{\ell,m=0}^{j} \|\hat{\nabla}^\ell \psi\|_4 \|\hat{\nabla}^{j-\ell}\psi\|_4 \|\hat{\nabla}^m \psi\|_4 \|\hat{\nabla}^{j-m}\psi\|_4 < \infty,$$

finally establishing that $q(\psi) \in L^{k,2}(i \wedge_+^2 T^*M)$.

Observe now that

$$Dq(\psi)\varphi = \overline{\varphi} \otimes \psi - \overline{\psi} \otimes \varphi - \operatorname{Re}\langle \varphi, \psi \rangle,$$

which is (real) linear on ψ. Hence, to see that q is smooth it suffices to check that Dq is continuous. This follows from

$$\|(Dq(\psi + \xi) - Dq(\psi))\varphi\| = \|\overline{\varphi} \otimes \xi + \overline{\xi} \otimes \varphi - \operatorname{Re}\langle \varphi, \xi \rangle\| \leq 3\|\xi\| \|\varphi\|. \qquad \square$$

Proposition 4.2.12. *Let $\eta \in L^{k,2}(\wedge^2 T^*M)$, for $k \geq 1$, be weakly closed ($d\eta = 0$ weakly), then the Seiberg–Witten map extends to a smooth map $\mathrm{SW}_\eta : \mathscr{C}^{k+1}(S) \to \mathscr{Y}^k(S)$.*

Proof. Identifying $\mathscr{A}^{k+1}(S) \cong L^{k+1,2}(iT^*M)$ by taking $\hat{\nabla}$ as a reference point, we can write $\mathrm{SW}_\eta : i\Omega^1(M) \times \Gamma(S^+) \to i\Omega_+^2(M) \times \Gamma(S^-)$ as

$$\mathrm{SW}_\eta(i\alpha, \psi) = (2i(d\alpha)_+ - q(\psi) + i\eta_+, \hat{\not{D}}\psi + i\alpha \cdot \psi).$$

Since both $\hat{\not{D}}$ and d are first order differential operators, they extend to smooth maps between Sobolev spaces of the correct regularity. The result now follows from Lemma 4.2.11 $\square$

We finish by proving that $\mathscr{G}^{k+2}(M)$ is a Hilber–Lie group, for which we will need the following result.

Theorem 4.2.13 (Sobolev multiplication)**.** *If $k, \ell \in \mathbb{N}$ are such that $k \geq \ell$ and $1 \leq p, q < \infty$ are such that*

$$\frac{1}{p} - \frac{k}{n} < 0 \qquad \text{and} \qquad \frac{1}{p} - \frac{k}{n} \leq \frac{1}{q} - \frac{\ell}{n},$$

then the multiplication of functions extends to a continuous map

$$L^{k,p}(M, \mathbb{C}) \times L^{\ell,q}(M, \mathbb{C}) \to L^{\ell,q}(M, \mathbb{C}).$$

Proposition 4.2.14. *For every $k \geq 1$, the group $\mathscr{G}^{k+2}(M)$ is a Hilbert–Lie group modeled on $L^{k+2,2}(M, i\mathbb{R})$, and the actions of $\mathscr{G}^{k+2}(M)$ on $\mathscr{C}^{k+1}(S)$ and $\mathscr{Y}^k(S)$ are smooth.*

Proof. First notice that by Theorem 4.2.9 we have that $\mathscr{G}^{k+2}(M) \subseteq C^0(M, \mathbb{C})$, since

$$\frac{1}{2} - \frac{k+2}{4} \leq -\frac{1}{4} < 0.$$

We can consider, then, the compact-open topology on $\mathscr{G}^{k+2}(M)$.

We will now provide charts for $\mathscr{G}^{k+2}(M)$. Consider first

$$\mathscr{H} := \{u \in \mathscr{G}^{k+2}(M) : u(M) \subseteq \mathrm{U}(1) \smallsetminus \{-1\}\},$$

which is an open neighborhood of the identity. The charts will be given by the Cayley transform as follows: consider the diffeomorphism $T : \mathbb{C} \to \mathbb{C}$ given by

$$T(z) = \frac{1-z}{1+z}.$$

It satisfies $T = T^{-1}$ and provides a diffeomorphism $i\mathbb{R} \to \mathrm{U}(1) \smallsetminus \{-1\}$. Consider now the chart around the identity $T : \mathscr{H} \to L^{k+2,2}(M, i\mathbb{R})$ given by $T(u) := T \circ u$, which is clearly a homeomorphism. Around any point $v \in \mathscr{G}^{k+2}(M)$, we define a chart $T_v : v\mathscr{H} \to L^{k+2,2}(M, i\mathbb{R})$ by $T_v(u) := T(v^{-1}u)$. Its inverse is given by $T_v^{-1}(if) = vT(if)$, so the transition functions $L^{k+2,2}(M, i\mathbb{R}) \to L^{k+2,2}(M, i\mathbb{R})$ are given by

$$T_u T_v^{-1}(if) = T(u^{-1} v T(if)),$$

which is smooth because T is so and the multiplication as well, by Sobolev multiplication (Theorem 4.2.13).

Finally, the multiplication in local charts T_u, T_v and T_w is expressed as

$$(if, ig) \longmapsto T(w^{-1} v T(if) u T(ig))$$

and inversion in local charts T_u and T_v as

$$if \longmapsto T(v^{-1} \overline{u T(if)}),$$

and both are smooth. □

Definition 4.2.15. We define $\mathscr{Z}_\eta^{k+1}(S)$ as $\mathrm{SW}_\eta^{-1}(0,0)$, for $\mathrm{SW}_\eta : \mathscr{C}^{k+1}(S) \to \mathscr{Y}^k(S)$, and $\mathscr{M}_\eta^{k+1}(S) := \mathscr{Z}_\eta^{k+1}(S)/\mathscr{G}^{k+2}(M)$.

Lecture 5

Smooth solutions and dimension of the moduli space

5.1 Elliptic operator theory

In the following, denote by $\mathbb{K}$ either $\mathbb{R}$ or $\mathbb{C}$. Let $E, F \to M$ be $\mathbb{K}$-vector bundles over a closed manifold M. Then we get a Fréchet topology on $\Gamma^\infty(E)$ and $\Gamma^\infty(F)$.

Definition 5.1.1. An **operator** P from E to F is a continuous $\mathbb{K}$-linear map $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$. The space of all operators from E to F is denoted $\mathrm{Op}(E, F)$.

P is called **local** if P preserves supports: for every $u \in \Gamma^\infty(E)$, $\mathrm{supp}(Pu) \subseteq \mathrm{supp}(u)$.

Remark 5.1.2. In the above definition, we assumed $\mathbb{K}$-linearity. If we drop this condition, we call P a **nonlinear operator**.

Local operators have the property that their behaviour is determined by how it looks in charts: if $P \in \mathrm{Op}(E, F)$ is local and U is an open subset, then for every $u_1, u_2 \in \Gamma^\infty(E)$ such that $u_1|_U = u_2|_U$, we have $(Pu_1)|_U = (Pu_2)|_U$, so $P|_U$ is a well defined operator. In particular, if (U, φ) is a trivialising chart[1] for $E, F \to M$, we see that we get a well defined operator

$$P_\varphi := \varphi \circ P_U \circ \varphi^{-1} : \Gamma^\infty(V \times \mathbb{K}^k) \to \Gamma^\infty(V \times \mathbb{K}^l).$$

Definition 5.1.3. Let $P \in \mathrm{Op}(E, F)$ be local and let $m \in \mathbb{N}_0$, then we say P is a **differential operator of order** $\leq m$ if for every trivialising chart (U, φ), $P_\varphi : \Gamma^\infty(V \times \mathbb{K}^k) \to \Gamma^\infty(V \times \mathbb{K}^l)$ is a matrix of differential operators of order $\leq m$, i.e.,

$$P_\varphi = \begin{pmatrix} P_{11} & \dots & P_{1k} \\ \vdots & \ddots & \vdots \\ P_{l1} & \dots & P_{lk} \end{pmatrix},$$

where $P_{ij} : C^\infty(V) \to C^\infty(V)$ is a (linear) differential operator of order $\leq m$ on V. We denote the space of differential operators of order $\leq m$ from E to F by $\mathrm{DO}_m(E, F)$.

The following is nontrivial:

[1] φ will simultaneously denote the maps $\varphi : U \to V \subseteq \mathbb{R}^n$, $\varphi : E|_U \to V \times \mathbb{K}^k$ and $\varphi : F|_U \to V \times \mathbb{K}^l$.

Theorem 5.1.4 (Peetre's theorem)**.** *Every local operator on a compact manifold is a differential operator of order* $\leq m$ *for some* $m \geq 0$ *.*

Now we turn to some examples:

Example 5.1.5. Let $M = S^1$ and let $E, F = M \times \mathbb{C}$. Then ∂_θ is a differential operator of order ≤ 1 from E to F .

Define $P : \Gamma^\infty(E) \to \Gamma^\infty(F)$ by

$$Pf(\theta) := \int_0^\theta (f(\theta') - \overline{f})d\theta' \,,$$

where $\overline{f}$ is the average value of f . Then P is not a local operator.

Example 5.1.6. Let $E, F \to M$ be $\mathbb{K}$-vector bundles, then

$$\mathrm{DO}_0(E, F) \cong \Gamma^\infty(\mathrm{Hom}(E, F)) \,.$$

If $E, F \cong M \times \mathbb{K}$,

$$\mathrm{DO}_1(E, F) \cong \mathfrak{X}^{\mathbb{K}}(M) \oplus \Gamma^\infty(M \times \mathbb{K}) \,,$$

where the first summand is the order 1 part and the second summand is the order 0 part.

In the above example, we see that DO_1 splits into an order 0 and a part of pure order 1. This is not true in general, we cannot split DO_m into DO_{m-1} and a part of pure order m : the Laplacian on $\mathbb{R}^2$ in Cartesian coordinates is

$$\Delta_{\mathbb{R}^2} = -\partial_x^2 - \partial_y^2 \,.$$

If we transform to polar coordinates, we get

$$\Delta_{\mathbb{R}^2} = -\partial_r^2 - \tfrac{1}{r^2}\partial_\theta^2 - \tfrac{1}{r}\partial_\theta \,,$$

so while it is of pure order 2 in Cartesian coordinates, it has an order 1 part in polar coordinates, i.e., the notion of pure order is not coordinate invariant.

What is true, however, is that the part of pure order m is tensorial, so $P \in \mathrm{DO}_m(E, F)$ always defines a section $P_m \in \Gamma^\infty(\mathrm{Sym}^m TM \otimes \mathrm{Hom}(E, F))$.

Definition 5.1.7. Let $E, F \to M$ be $\mathbb{C}$-vector bundles and let $\pi : T^*M \to M$ be the real cotangent bundle. The **symbol map** $\sigma_m : \mathrm{DO}_m(E, F) \to \Gamma^\infty(\mathrm{Hom}(\pi^*E, \pi^*F))$ is defined by

$$\sigma_m(P) := i^m P_m \,,$$

interpreted as a $\mathrm{Hom}(E, F)$-valued homogeneous polynomial order m on T^*M .

Remark 5.1.8. The above definition only makes sense for $\mathbb{C}$-vector bundles because of the i^m in the definition. If we want the same definition for $\mathbb{R}$-vector bundles, we could define $\sigma_m(P) = P_k$, but since the above definition works better for complex vector bundles, and since we will really only complex vector bundles in the remainder, we will restrict to the case $\mathbb{K} = \mathbb{C}$ from now on.

Example 5.1.9. The following are the symbols of some well-known differential operators:

1. Let $\Delta_{\mathbb{R}^n}$ be the Laplacian on $\mathbb{R}^n$. Then $\sigma_2(\Delta_{\mathbb{R}^n}) = \sum_{i=1}^n (\xi_i)^2$, where ξ_i denote the Cartesian coordinates on the fibres of $T^*\mathbb{R}^n \cong \mathbb{R}^n_{(x_1,\dots,x_n)} \times \mathbb{R}^n_{(\xi_1,\dots,\xi_n)}$.

2. Let $X \in \mathfrak{X}^{\mathbb{C}}(M)$ interpreted as a differential operator of order ≤ 1 from $M \times \mathbb{C}$ to $M \times \mathbb{C}$. Then $\sigma_1(X) = iX \in C^\infty_{\mathrm{lin}}(T^*M)$.

3. Let $\nabla : \Gamma^\infty(E) \to \Gamma^\infty(T^*M^{\mathbb{C}} \otimes E)$ be a connection on a $\mathbb{C}$-vector bundle E. Then $\nabla \in \mathrm{DO}_1(E, T^*M^{\mathbb{C}} \otimes E)$ and

$$\sigma_1(\nabla)(\xi) = i\xi \otimes - \in \mathrm{Hom}(\pi^*E, \pi^*(T^*M^{\mathbb{C}} \otimes E)).$$

4. By antisymmetrising the previous example, we obtain

$$\sigma_1(d)(\xi) = i\xi \wedge - \in \mathrm{Hom}(\pi^*\Lambda^k T^*M^{\mathbb{C}}, \pi^*\Lambda^{k+1}T^*M^{\mathbb{C}}).$$

The following is not too difficult to prove:

Proposition 5.1.10. *Let $E_1, E_2, E_3 \to M$ be complex vector bundles and let $P \in \mathrm{DO}_m(E_1, E_2)$ and $Q \in \mathrm{DO}_{m'}(E_2, E_3)$ for some $m, m' \in \mathbb{N}_0$.*

1. *If $m \neq 0$, $\sigma_m(P)|_{0_{T^*M}} = 0$.*

2. *If $\sigma_m(P) \equiv 0$, then $P \in \mathrm{DO}_{m-1}(E, F)$.*

3. *$Q \circ P \in \mathrm{DO}_{m+m'}$ and $\sigma_{m+m'}(Q \circ P) = \sigma_{m'}(Q) \circ \sigma_m(P)$.*

4. *If M is Riemannian and E_1, E_2 are hermitian, the formal adjoint $P^* : \Gamma^\infty(E_2) \to \Gamma^\infty(E_1)$ defined implicitly by*

$$\int_M \langle Pu, v\rangle_F \,\mathrm{vol} = \int_M \langle u, P^*v\rangle_E \,\mathrm{vol}; \qquad u \in \Gamma^\infty_c(E_1), v \in \Gamma^\infty(E_2),$$

is well defined and satisfies $P^* \in \mathrm{DO}_m(E_2, E_1)$ and $\sigma_m(P^*) = (\sigma_m(P))^*$.

Definition 5.1.11. Let $E, F \to M$ be complex vector bundles and let $P \in \mathrm{DO}_m(E, F)$. Then P is **elliptic** if $\sigma_m(P)(\xi) \in \mathrm{Hom}(E_{\pi(\xi)}, F_{\pi(\xi)})$ is invertible whenever $\xi \neq 0$. Denote by $\mathrm{DO}^{\mathrm{ell}}_m(E, F)$ the space of all elliptic operators from E to F.

If $E_0, \dots, E_l \to M$ are complex vector bundles and $P_i \in \mathrm{DO}_{m_i}(E_i, E_{i+1})$ for $i = 1, \dots, l-1$ are such that $P_i \circ P_{i-1} = 0$ for every $i = 1, \dots, l-1$, then the complex

$$\Gamma^\infty(E_0) \xrightarrow{P_0} \Gamma^\infty(E_1) \xrightarrow{P_1} \dots \xrightarrow{P_{l-1}} \Gamma^\infty(E_{l-1})$$

is **elliptic** if the associated **symbol sequence**

$$0 \longrightarrow E_{0,\pi(\xi)} \xrightarrow{\sigma_{m_0}(P_0)(\xi)} E_{1,\pi(\xi)} \xrightarrow{\sigma_{m_1}(P_1)(\xi)} \dots \xrightarrow{\sigma_{m_{l-1}}(P_{l-1})(\xi)} E_{l-1,\pi(\xi)} \longrightarrow 0$$

is exact whenever $\xi \neq 0$.

The following facts are fundamental facts for the analysis of differential operators. From now on, for simplicity, we will implicitly assume M is compact, although sometimes one can get away with picking a Riemannian metric on M and hermitian metrics on every relevant vector bundle.

Theorem 5.1.12. *Let $E, F \to M$ be complex vector bundles and let $P \in \mathrm{DO}_m(E,F)$. Then for every $k \in \mathbb{N}_0$ and $p \in [1,\infty)$, P extends canonically to a continuous linear map $P : L^{k+m,p}(E) \to L^{k,p}(F)$.*

Theorem 5.1.13. *Let $E, F \to M$ be complex vector bundles and let $P \in \mathrm{DO}^{\ell}_m(E,F)$. Then there is an operator $Q \in \mathrm{OP}(F,E)$ such that for every $k \in \mathbb{N}_0$ and $p \in [1,\infty)$, Q extends to a continuous linear map $Q : L^{k,p}(F) \to L^{k+m,p}(E)$ such that $\mathrm{id} - Q \circ P : L^{k+m,p}(E) \to L^{k+m,p}(E)$ lands in $L^{k+m+1,p}(E)$ and $\mathrm{id} - P \circ Q : L^{k,p}(F) \to L^{k,p}(F)$ lands in $L^{k+1,p}(F)$.*

Definition 5.1.14. The operator $Q \in \mathrm{OP}(E,F)$ is called a *weak-inverse* to P.

Remark 5.1.15. An elliptic operator usually doesn't have a unique weak inverse. In fact, a weak inverse can always be chosen such that $\mathrm{id} - QP$ and $\mathrm{id} - PQ$ land in Γ^∞. Moreover, Q is not local, so it is not a differential operator, it fits into the theory of *pseudodifferential operators*.

Example 5.1.16. Let $M = S^1$ and $E, F = M \times \mathbb{C}$. Then the operator ∂_θ is elliptic with weak inverse

$$Q(f)(\theta) = \int_0^\theta (f(\theta') - \overline{f})d\theta' ,$$

where $\overline{f}$ is the average value of f.

From this point onwards, we need compactness everywhere, one can no longer get away with picking metrics. One nice thing about elliptic operators (on compact manifolds) is that they are Fredholm[2].

Theorem 5.1.17. *Let $E, F \to M$ be complex vector bundles and let $P \in \mathrm{DO}^{\mathrm{ell}}_m(E,F)$. Then for every $k \in \mathbb{N}_0$ and $p \in [1,\infty)$, $P : L^{k+m,p}(E) \to L^{k,p}(F)$ is a Fredholm operator.*

Proof. Since P is elliptic, it has weak-inverse $Q \in \mathrm{OP}(F,E)$. Then $\mathrm{id} - PQ : L^{k,p}(F) \to L^{k,p}(E)$ lands in $L^{k+1,p}(E)$, but by Rellich-Kondrachov, the embedding $L^{k+1,p}(E) \hookrightarrow L^{k,p}(E)$ is compact, so $\mathrm{id} - PQ : L^{k,p}(E) \to L^{k,p}(E)$ is a compact operator. Likewise, $\mathrm{id} - QP : L^{k+m,p}(F) \to L^{k+m,p}(F)$ is compact. Thus, P is invertible up to a compact operator, so P is Fredholm. □

Theorem 5.1.18. *The Fredholm index[3] of $P \in \mathrm{DO}^{\mathrm{ell}}_m(E,F)$ depends only on topological properties of $\sigma_m(P)$. Moreover, the index of P is independent of k and p.*

Remark 5.1.19. The above theorem is stated in rather vague terms. The precise statement here is known as the *Atiyah-Singer index theorem.* This theorem also gives a precise formula for this index in terms of a K-theory class associated to $\sigma_M(P)$.

Example 5.1.20. The following are examples of symbols of certain elliptic operators

1. Let $X \in \mathfrak{X}^{\mathbb{R}}(S^1)$ be nonvanishing, and interpret X as an order ≤ 1 differential operator on the trivial line bundle $S^1 \times \mathbb{C}$. Then X is elliptic and it can be deformed into ∂_θ through elliptic operators, so $\mathrm{index}(X) = \mathrm{index}(\partial_\theta) = 0$, since

$$\ker(\partial_\theta) = \mathrm{coker}(\partial_\theta) = \{\text{constant functions}\} .$$

[2]Recall that a Fredholm operator is a linear operator with a finite dimensional kernel and cokernel.
[3]Recall that the Fredholm index is $\dim(\ker(P)) - \dim(\mathrm{coker}(P))$

2. The following can be computed using the full Atiyah-Singer index theorem: let (M^4, s) be a spin^c manifold with spinor bundle S. Then

$$\operatorname{index}(\not{D}_S) = \frac{c_1(\det(s))^2 - \sigma(M)}{8},$$

where $\sigma(M)$ is the signature of M.

3. Let M be a Riemannian manifold, then $d + d^* : \Omega^{\text{even}}(M) \to \Omega^{\text{odd}}(M)$ is a Fredholm operator with $\operatorname{index}(d + d^*) = \chi(M)$, which is a consequence of the Hodge decomposition theorem.

Another nice thing about elliptic operators is that they behave extremely well with respect to regularity:

Theorem 5.1.21 (Elliptic regularity)**.** *Let $P \in \mathrm{DO}_m^{\mathrm{ell}}(E, F)$, $u \in L^{m,p}(E)$ such that $Pu \in L^{k,p}(F)$, then $u \in L^{k+m,p}(E)$.*

Proof. Let Q be a weak inverse to P. Since $Pu \in L^{k,p}(F)$, we see $QPu \in L^{k+m,p}(E)$. We also know $u - QPu \in L^{m+1,p}(E)$, so if $k \geq 1$, we conclude $u \in L^{m+1,p}(E)$, but then $u - QPu \in L^{m+2,p}(E)$, so if $k \geq 2$, we conclude $u \in L^{m+2,p}(E)$. Iterating this k times, we find $u \in L^{m+k,p}(E)$. □

Remark 5.1.22. The above proof technique is known as *bootstrapping*: one cannot pull themself up by their own bootstraps, but a solution to an elliptic equation can pull its own regularity up by using its own regularity.

Corollary 5.1.23. *Let $m \geq 1$, $P \in \mathrm{DO}_m^l(E, E)$ and let $u \in L^{m,p}(E)$ be an eigenfunction of P. Then u is smooth.*

Proof. Let $\lambda \in \mathbb{C}$ be the corresponding eigenvalue of u. Then the operator $P - \lambda\mathrm{id}$ is elliptic, as its degree m part agrees with the degree m part of P, so $\sigma_m(P - \lambda\mathrm{id}) = \sigma_m(P)$. But $Pu - \lambda u = 0$, and 0 is smooth, so u is $L^{k,p}$ for any k, so by Morrey's theorem, u is smooth. □

5.2 Applications to the Seiberg-Witten moduli space

For this section, we let (M^4, s) be a spin^c manifold with spinor bundle S and we pick a smooth reference connection $\hat{\nabla}$ on $\det(s)$.

5.2.1 Smoothness of solutions

Theorem 5.2.1. *Let $\eta \in \Omega^2_+(M)$ and $(i\alpha, \psi) \in \mathscr{Z}^2_\eta(S)$. Then there is a $u \in \mathscr{G}^3(M)$ such that $u \cdot (i\alpha, \psi)$ is smooth. I.e., every solution to the Seiberg-Witten equations with regularity $L^{2,2}$ can be gauge-transformed into a smooth solution.*

Proof. Using the Hodge decomposition theorem, we write $i\alpha = i\alpha_0 + idf + id^*\beta$, where α_0 is harmonic, $f \in L^{3,2}(M)$ and $\beta \in L^{3,2}(\Lambda^2 T^* M)$. Pick $u = e^{if} \in \mathscr{G}^3$, then $u \cdot (i\alpha) = (i\alpha_0 + id^*\beta, e^{if}\psi) =: (i\widetilde{\alpha}, \widetilde{\psi})$.

Now we will use the Seiberg-Witten equations to bootstrap the regularity of $(i\widetilde{\alpha}, \widetilde{\psi})$ then tell us

$$\hat{\not{D}}\widetilde{\psi} = -\widetilde{\alpha} \cdot \widetilde{\psi}$$

$$2id^+\widetilde{\alpha} = q(\widetilde{\psi}) - i\eta - F^+_{\hat{\nabla}}\,.$$

Since $\widetilde{\alpha}, \widetilde{\psi}$ have regularity $L^{2,2}$, the Sobolev embedding theorem imply they are L^p for any $p \in [1,\infty)$, so the Hölder inequality tells us $i\widetilde{\alpha}\cdot\widetilde{\psi} \in L^p$ for any p. Thus, the first equation implies $\hat{\not{D}}\widetilde{\psi}$ is L^p, so ellipticity of the Dirac operator and the elliptic regularity theorem, we conclude $\widetilde{\psi}$ is $L^{1,p}$, so the Hölder inequality tells us $q(\widetilde{\psi})$ is also $L^{1,p}$, so $d^+\widetilde{\alpha}$ is also $L^{1,p}$.

Now, $d^+ + d^* : \Omega^1 \to \Omega^2_+ \oplus \Omega^0$ is an elliptic operator[4], and $\widetilde{\alpha}$ is coclosed, so $d^+\alpha = (d^+ + d^*)\widetilde{\alpha}$, so elliptic regularity implies $\widetilde{\alpha}$ is $L^{2,p}$. Now we keep bootstrapping to find $(i\widetilde{\alpha}, \widetilde{\psi})$ is smooth, completing the proof. □

Corollary 5.2.2. *Let $k, l \in \mathbb{N}$, then $\mathscr{M}^{k+1}_\eta(S) \cong \mathscr{M}^{l+1}_\eta(S)$. I.e., the moduli space is independent of k.*

5.2.2 Expected dimension of the moduli space

At this point, we know nothing about the moduli space, but we can compute the dimension of the Zariski tangent space of $\mathscr{M}^2_\eta$ by linearising the Seiberg-Witten operator around a point in $\mathscr{Z}^2_\eta(S)$, computing the kernel, and quotienting out the image of the infinitesimal action of the gauge group.

Lemma 5.2.3. *Let $(i\alpha,\psi) \in \mathscr{Z}^2_\eta(S)$, then:*

1. *the infinitesimal action $\mathfrak{a}^2|_{(i\alpha,\psi)} : L^{3,2}(M, i\mathbb{R}) \to \mathscr{C}^2_\eta(S) = L^{2,2}(iT^*M \oplus S^+)$ is given by*

$$\mathfrak{a}^2|_{(i\alpha,\psi)}(if) = (-idf, if\psi)\,; \tag{5.1}$$

2. *the derivative $D\,SW_\eta|_{(i\alpha,\psi)} : L^{2,2}(iT^*M \oplus S^+) \to L^{1,2}(i\Lambda^2_+T^*M \oplus S^-)$ is given by*

$$D\,SW_\eta|_{(i\alpha,\psi)}(i\beta,\varphi) = (2id^+\beta - Dq|_\psi(\varphi), \hat{\not{D}}\varphi + i\alpha\cdot\varphi + i\beta\cdot\psi)\,, \tag{5.2}$$

where $Dq|_\psi(\varphi) = \overline{\varphi}\otimes\psi + \overline{\psi}\otimes\varphi - (\langle\varphi,\psi\rangle + \overline{\langle\varphi,\psi\rangle})/2$.

Proof. 1. Let $f \in L^{3,2}(M, i\mathbb{R})$. Then we compute

$$\begin{aligned}\mathfrak{a}^2|_{(i\alpha,\psi)}(if) &= \frac{d}{dt}\bigg|_{t=0} e^{ift}\cdot(i\alpha,\psi)\\ &= \frac{d}{dt}\bigg|_{t=0} (i\alpha - tidf, e^{ift}\psi)\\ &= (-idf, if\psi)\,.\end{aligned}$$

2. Likewise, let $(i\beta,\varphi) \in L^{2,2}(iT^*M \oplus S^+)$. Then we compute

$$\begin{aligned}D\,SW_\eta|_{(i\alpha,\psi)}(i\beta,\varphi) &= \frac{d}{dt}\bigg|_{t=0} SW_\eta(i\alpha + ti\beta, \psi + t\varphi)\\ &= (2id^+\alpha + 2tid^+\beta - q(\psi + t\varphi) + i\eta, \hat{\not{D}}(\psi + t\varphi) + (i\alpha + ti\beta)\cdot(\psi + t\varphi))\\ &= (2id^+\beta - Dq|_\psi(\varphi), \hat{\not{D}}\varphi + i\alpha\cdot\varphi + i\beta\cdot\psi)\,.\end{aligned}$$

□

[4] To prove this, note that $\ker(\sigma_1(d^+)(\xi)) = \{\lambda\xi\}$ for nonzero $\xi \in T^*M$, but $\sigma_1(d^*)(\xi)(\lambda\xi) = \lambda\|\xi\|^2$, so the symbol of $d^+ + d^* : \Omega^1 \to \Omega^2_+ \oplus \Omega^0$ is injective.

Theorem 5.2.4. *Let* $(i\alpha,\psi)\in\mathscr{Z}^2_\eta(S)$, *then the sequence* $(SW^\bullet_\eta(i\alpha,\psi),d)$ *defined as*

$$0\longrightarrow L^{3,2}(M,i\mathbb{R})\xrightarrow{\mathfrak{a}^2|_{(i\alpha,\psi)}} L^{2,2}(iT^*M\oplus S^+)\xrightarrow{D\,SW_\eta|_{(i\alpha,\psi)}} L^{1,2}(i\Lambda^2_+T^*M\oplus S^-)\longrightarrow 0$$

is an elliptic complex. I.e.,

$$D\,SW_\eta|_{(i\alpha,\psi)}\circ\mathfrak{a}^2|_{(i\alpha,\psi)}=0\,,$$

and the associated symbol sequence

$$0\longrightarrow \pi^*i\mathbb{R}\xrightarrow{\sigma_1(\mathfrak{a}^2|_{(i\alpha,\psi)})}\pi^*(iT^*M\oplus S^+)\xrightarrow{\sigma_1(D\,SW_\eta|_{(i\alpha,\psi)})}\pi^*(i\Lambda^2_+T^*M\oplus S^-)\longrightarrow 0$$

is exact. Moreover, the real Euler characteristic satisfies

$$\chi(SW^\bullet_\eta(i\alpha,\psi),d)=\frac{2\chi(M)+3\sigma(M)-c_1(\det s)^2}{4}\,.$$

Proof. Most computations here are straightforward, so we will focus on the computation of the Euler characteristic. To compute that one, note that the Euler characteristic only depends on the principal part of the elliptic complex. The principal parts form the following elliptic complex

$$0\longrightarrow L^{3,2}(M,i\mathbb{R})\xrightarrow{(d,0)} L^{2,2}(iT^*M\oplus S^+)\xrightarrow{2d^+\oplus\hat{\not{D}}} L^{1,2}(i\Lambda^2_+T^*M\oplus S^-)\longrightarrow 0\,,$$

which splits into a direct sum of the following sequences

$$0\longrightarrow L^{3,2}(M,i\mathbb{R})\xrightarrow{(d,0)} L^{2,2}(iT^*M)\xrightarrow{2d^+} L^{1,2}(i\Lambda^2_+T^*M)\longrightarrow 0$$

$$0\longrightarrow 0\longrightarrow L^{2,2}(S^+)\xrightarrow{\hat{\not{D}}} L^{1,2}(S^+)\longrightarrow 0\,.$$

The Euler characteristic of the top sequence is $1-b_1+b_2^+=(\chi(M)+\sigma(M))/2$, whereas the real Euler characteristic of the bottom sequence is $(\sigma(M)-c_1(\det s)^2)/4$, as given in Example 5.1.20 2, completing the proof. ☐

If we add the observation that the gauge group acts freely on irreducible solutions, we obtain

Corollary 5.2.5. *Let* $(i\alpha,\psi)\in\mathscr{Z}^2_\eta(S)$ *be irreducible. Then the dimension of the Zariski tangent space* $T_{(i\alpha,\psi)}\mathscr{M}_\eta$ *is*

$$\frac{-2\chi(M)-3\sigma(M)+c_1(\det s)^2}{4}+\dim(H^2(SW^\bullet_\eta,d))\,. \tag{5.3}$$

Lecture 6

Compactness of the moduli space

In section 4.2.3, we introduced the spaces on which we study the Seiberg-Witten equations. The goal of this chapter is to show that the moduli space $\mathscr{M}_\eta^{k+1}$ introduced on page 29 is a Hausdorff and compact topological space.

The gauge action by $\mathscr{G}^{k+2}$ on $\mathscr{C}^{k+1}$ gives rise to a quotient space, which we denote by $\mathscr{B}^{k+1} := \mathscr{C}^{k+1}/\mathscr{G}^{k+2}$. Then we have $\mathscr{M}_\eta^{k+1} \subset \mathscr{B}^{k+1}$ as a topological subspace. Therefore, it is sufficient to show that $\mathscr{B}^{k+1}$ is Hausdorff to conclude that $\mathscr{M}_\eta^{k+1}$ is Hausdorff, and this is precisely what we will do.

Then after we have established Hausdorffness, we will establish *a priori* bounds for the solutions of the Seiberg-Witten equations. These bounds then allow us to show that $\mathscr{M}_\eta^{k+1}$ is also compact. Moreover, using these same bounds, we will show that given a Riemannian manifold and a perturbation parameter η, there are only finitely many spin^c structures that admit admit solutions *and* have non-negative formal dimension.

The contents of this chapter are mainly based on section 2.2.1 of [Nic00] and chapter 4 and 5 of [Mor96].

6.1 Multiplication of functions in Sobolev spaces

In the course of this chapter we will regularly need to multiply functions that live in Sobolev spaces. However, in general there is no reason for these products to be an element of a Sobolev space again. The following theorem gives us conditions under which the product of functions in certain Sobolev spaces are again in a Sobolev space.

Theorem 6.1.1 (Theorem 7.3 in [BH21])**.** *Assume that s_i, s are natural numbers and $1 \leq p_i \leq p < \infty$ $(i = 1, 2)$ are real numbers satisfying*

(i) $s_i \geq s$

(ii) $s \geq 0$

(iii) $s_i - s \geq n\left(\frac{1}{p_i} - \frac{1}{p}\right)$,

(iv) $s_1 + s_2 - s > n\left(\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p}\right)$.

Then the map

$$L^{s_1,p_1} \times L^{s_2,p_2} \to L^{s,p}, (f, g) \mapsto fg$$

is continuous bilinear.

From this we deduce the following results, which we record in separate lemmas to be used later in our proofs. The proof of each of these lemmas is checking that conditions 1–2 in the theorem above are satisfied.

Lemma 6.1.2. *Let k, ℓ be integers. If $k \geq 3$ and $k \geq \ell$, then the map*

$$L^{k,2} \times L^{\ell,2} \to L^{\ell,2}, (f,g) \mapsto fg$$

is continuous bilinear.

Lemma 6.1.3. *Let $p > 2$. Then the map*

$$L^{2,2} \times L^{2,p} \to L^{2,2}, (f,g) \mapsto fg$$

is continuous bilinear.

Lemma 6.1.4. *Let $1 \leq p \leq 4$ and let $p < q$. Then the map*

$$L^{2,2} \times L^{1,q} \to L^{1,p}, (f,g) \mapsto fg$$

is continuous bilinear.

Lemma 6.1.5. *Let $1 \leq q < p < \infty$. Then the map*

$$L^{2,2} \times L^{p} \to L^{q}, (f,g) \mapsto fg$$

is continuous bilinear.

Proof. From the Sobolev Embedding Theorem, theorem 4.2.9, it follows that $L^{2,2} \hookrightarrow L^r$ for all $1 \leq r < \infty$. The result now follows by the Hölder inequality. □

6.2 Hausdorffness of the quotient space

In this section, we will show that $\mathscr{B}^{k+1}$ is a Hausdorff space. We do this by showing that the action of the gauge group $\mathscr{G}^{k+2}$ on $\mathscr{C}^{k+1}$ is proper. For more details on proper group actions in general, see appendix A.

Proposition 6.2.1. *Let (ψ_n, A_n) and (μ_n, B_n) be sequences in $\mathscr{C}^{k+1}$ converging to (ψ, A) and (μ, B) respectively. Suppose that for each n we have a $\gamma_n \in \mathscr{G}^{k+2}$ such that*

$$\gamma_n \cdot (\psi_n, A_n) = (\mu_n, B_n).$$

Then there is a subsequence of (γ_n) which converges to $\gamma \in \mathscr{G}^{k+2}$. Moreover, we have

$$\gamma \cdot (\psi, A) = (\mu, B).$$

Proof. Since $\gamma_n \cdot (\psi_n, A_n) = (\mu_n, B_n)$, we have

$$d\gamma_n = \frac{1}{2}\gamma_n(B_n - A_n).$$

We note that the operator $d + d^* : \Omega^{\text{even}} \to \Omega^{\text{odd}}$ is elliptic and that for functions $(d+d^*)\gamma = d\gamma$.

We make a case distinction between $k = 1$ and $k \geq 2$.

Case $k = 1$: We have $\gamma_n \in L^{3,2}$ so by Morrey's theorem, theorem 4.2.9, all $\gamma_n \in C^0$. Then we obtain a bound

$$\|d\gamma_n\|_{L^q} \leq C\|\gamma_n\|_{C^0}\|B_n - A_n\|_{L^q}$$

for all $1 \leq q < \infty$. By elliptic regularity, theorem 5.1.21, it follows that $\gamma_n \in L^{1,q}$ for all $1 \leq q < \infty$. Then by lemma 6.1.4 we have

$$\|d\gamma_n\|_{L^{1,p}} \leq C\|\gamma_n\|_{L^{1,q}}\|B_n - A_n\|_{L^{2,2}}$$

for $1 \leq p \leq 4$ and $q > p$. So $d\gamma_n \in L^{1,p}$ for $1 \leq p \leq 4$ and so by ellipticity we have $\gamma_n \in L^{2,4}$. We have $\frac{1}{4} - \frac{1}{2} < \frac{1}{6} - \frac{1}{4}$, so by the Rellich-Kondrachov theorem, theorem 4.2.9, we have a compact embedding $L^{2,4} \hookrightarrow L^{1,6}$. So there is a $L^{1,6}$-convergent subsequence of γ_n, which we will denote still by γ_n. It converges to some $\gamma \in L^{1,6}$. By Morrey's theorem, theorem 4.2.9, $L^{1,6} \hookrightarrow C^0$, so this subsequence also converges in the C^0 sense. In particular we see that $|\gamma| = 1$ everywhere.

We have $d\gamma_n \to d\gamma$ in L^6. By lemma 6.1.5 we obtain that $\frac{1}{2}\gamma_n(B_n - A_n) \to \frac{1}{2}\gamma(B - A)$ in L^6 as well. So $d\gamma = \frac{1}{2}\gamma(B - A)$. Then by lemma 6.1.4 we see that $\frac{1}{2}\gamma(B - A)$ is in $L^{1,p}$ for $1 \leq p \leq 4$. So by ellipticity, $\gamma \in L^{2,4}$. Then by lemma 6.1.3 we see that $\frac{1}{2}\gamma(B - A)$ is in $L^{2,2}$, so by ellipticity $\gamma \in L^{3,2}$. So $\gamma \in \mathscr{G}^{k+2}$. Finally, we have $\gamma_n\psi_n \to \gamma\psi$ by lemma 6.1.2 and so $\gamma \cdot (\psi, A) = (\mu, B)$.

Case $k \geq 2$: Then we have $\gamma_n \in L^{k+2,2}$. By the Rellich-Kondrachov theorem, theorem 4.2.9, this embeds compactly in $L^{k+1,2}$, so we have a convergent subsequence in $L^{k+1,2}$, which we also denote by γ_n. Then as above, we obtain that $|\gamma| = 1$ everywhere and $d\gamma = \frac{1}{2}\gamma(B - A)$. Then by lemma 6.1.2 we see that $d\gamma \in L^{k+1,2}$, so by ellipticity $\gamma \in L^{k+2,2}$. So $\gamma \in \mathscr{G}^{k+2}$. The final assertion again follows from lemma 6.1.2. □

Remark 6.2.2. This proposition can be generalised to nets of configurations and gauge group elements. This would yield another proof of the Hausdorffness, where one can circumvent appealing to first countability of the quotient space.

This proposition implies the following.

Corollary 6.2.3. *The action $\mathscr{G}^{k+2} \curvearrowright \mathscr{C}^{k+1}$ is proper.*

Proof. Let $(\psi_n, A_n) \subset \mathscr{C}^{k+1}$ a sequence and $\gamma_n \subset \mathscr{G}^{k+2}$ a sequence. Assume that $(\psi_n, A_n) \to (\psi, A)$ and that $\gamma_n \cdot (\psi_n, A_n) \to (\mu, B)$. Then defining $(\mu_n, B_n) = \gamma_n \cdot (\psi_n, A_n)$ we see that we satisfy the condition of the theorem, so γ_n has a convergent subsequence. Hence the action is proper, by proposition A.0.2. □

This implies the Hausdorffness we claimed.

Corollary 6.2.4. *The quotient space $\mathscr{B}^{k+1} = \mathscr{C}^{k+1}/\mathscr{G}^{k+2}$ is Hausdorff.*

Proof. By corollary 6.2.3 the group action is proper. So by proposition A.0.6, the quotient $\mathscr{B}^{k+1} = \mathscr{C}^{k+1}/\mathscr{G}^{k+2}$ is Hausdorff. □

Corollary 6.2.5. *The Seiberg-Witten moduli space $\mathscr{M}_\eta^{k+1}$ is Hausdorff.*

Proof. The moduli space is a subspace of $\mathscr{B}^{k+1}$. □

6.3 Curvature bounds

Our next goal is to show that the moduli space is compact. For this we need to estimate the different terms in the Seiberg-Witten equations. In this section we will establish some of the necessary bounds.

We start by estimating the norm of the Clifford action γ.

Lemma 6.3.1. *Let $\alpha \in i\Omega^2_+(M)$. Then the we have the pointwise identity*

$$|\gamma_\alpha|^2 = 4|\alpha|^2,$$

using the Frobenius norm.

Proof. This identity holds pointwise, so it suffices to check it in a point $x \in M$. The Frobenius norm is given by $|\gamma_\alpha|^2 = \mathrm{tr}(\gamma_\alpha^* \gamma_\alpha)$. Since α is in $i\Omega^2_+(M)$, we have $\gamma_\alpha^* = -\gamma_\alpha$. There is a orthonormal basis $\{\eta_0, \eta_1, \eta_2\}$ for $\wedge^2_+ T^*_x M$ given by

$$\begin{aligned}
\eta_0 &= \frac{1}{\sqrt{2}}\left(dx^1 \wedge dx^2 + dx^3 \wedge dx^4\right)\\
\eta_1 &= \frac{1}{\sqrt{2}}\left(dx^1 \wedge dx^3 - dx^2 \wedge dx^4\right)\\
\eta_2 &= \frac{1}{\sqrt{2}}\left(dx^1 \wedge dx^4 + dx^2 \wedge dx^3\right),
\end{aligned}$$

where (x^i) are normal coordinates at x.

A computation shows that this basis has the property that $\gamma(\eta_k)^2 = -2\mathrm{id}$ and $\gamma(\eta_k)\gamma(\eta_l) = 0$ for $k \neq l$ at x. So if we write $\alpha = \sum_{k=0}^2 \alpha_k \eta_k$, then we have with Clifford multiplication that

$$\gamma(\alpha)^2 = \gamma\left(\sum_k \alpha_k \eta_k\right)^2 = -2\left(\sum_{k=0}^2 \alpha_k^2\right)\mathrm{id}.$$

This identity map lives on a (complex) dimension 2 space, so $\mathrm{tr}(\mathrm{id}) = 2$. Hence the norm $|\gamma(\alpha)|^2 = 4|\alpha|^2$. □

Next, we have the following equation for solutions of the unperturbed Seiberg-Witten equation.

Lemma 6.3.2. *Let (ψ, A) be a solution to the unperturbed Seiberg-Witten equations on a compact four-manifold M. Then*

$$\|\nabla(\psi)\|^2_{L^2} + \frac{1}{4}\langle \mathrm{scal}\,\psi, \psi\rangle_{L^2} + \frac{\|\psi\|^4_{L^4}}{4} = 0.$$

Here scal *denotes the scalar curvature of M.*

Proof. Since (ψ, A) is a solution to the Seiberg-Witten equations, we have $\not{D}\psi = 0$, so by the Weitzenböck formula, theorem 4.1.20, we have

$$0 = \not{D}^2\psi = \nabla^*\nabla\psi + \frac{\mathrm{scal}}{4}\psi + \frac{F_A}{2}\cdot\psi.$$

Using the Seiberg-Witten equations, we can therefore write

$$0 = \nabla^*\nabla\psi + \frac{\mathrm{scal}}{4}\psi + \frac{1}{2}\left(\overline{\psi}\otimes\psi - \frac{|\psi|^2}{2}\mathrm{id}\right)\psi$$

$$= \nabla^*\nabla\psi + \frac{\text{scal}}{4}\psi + \frac{|\psi|^2}{4}\psi.$$

Taking the L^2-inner product with ψ then yields the asserted equality. ☐

This implies the following result for the non-existence of irreducible spinors.

Corollary 6.3.3. *Let M be a compact four-manifold. Let $\kappa_M^- = \max_{x\in M}(0, -\text{scal}(x))$. Then*

$$\kappa_M^- \|\psi\|_{L^2}^2 \geq \|\psi\|_{L^4}^4.$$

In particular, if M has non-negative scalar curvature, then any solution to the unperturbed Seiberg-Witten equations has trivial spinor field, i.e., it is reducible.

Proof. We have for all $x \in M$ that $-\text{scal}(x) \leq \kappa_M^-$, so $\text{scal}(x) \geq -\kappa_M^-$. Therefore

$$\langle \text{scal}\psi, \psi\rangle = \int_M \text{scal}|\psi|^2 \text{vol}_g \geq -\kappa_M^- \|\psi\|_{L^2}^2.$$

Using the equality from the previous lemma, we obtain

$$\frac{1}{4}\|\psi\|_{L^4}^4 \leq \|\nabla\psi\|_{L^2}^2 + \frac{1}{4}\|\psi\|_{L^4}^4 = -\frac{1}{4}\langle \text{scal}\psi, \psi\rangle \leq \frac{1}{4}\kappa_M^- \|\psi\|_{L^2}^2.$$

The final assertions follows by noting that for a manifold with non-negative scalar curvature $\kappa_M^- = 0$. ☐

Next, we will derive a pointwise estimate for the spinor. To do this we need the following lemma's about the gradient and the Laplace-Beltrami operator.

Lemma 6.3.4. *Let (M, g) be an n-dimensional Riemannian manifold. Let $f \in C^\infty(M)$ and let (E_i) be a local orthonormal frame of TM. Then the gradient of f is locally given by*

$$\text{grad}(f) = \sum_{i=1}^n E_i(f)E_i.$$

Proof. Let (ε^j) denote the orthonormal frame of T^*M dual to (E_i). Then we have $\varepsilon^j(E_i) = \delta_i^j$, so

$$df = \sum_{j=1}^n df(E_j)\varepsilon^j = \sum_{j=1}^n E_j(f)\varepsilon^j.$$

Since the frame ε^j is orthonormal dual to E_i, we have $(\varepsilon^j)^\sharp = E_j$. So

$$\text{grad}(f) = (df)^\sharp = \sum_{j=1}^n E_j(f)(\varepsilon^j)^\sharp = \sum_{j=1}^n E_j(f)E_j.$$

☐

Recall that the Laplace-Beltrami operator is defined as $\Delta\colon C^\infty(M) \to C^\infty(M)$,

$$\Delta(f) = -\text{div}(\text{grad}(f)).$$

We then have the following local form.

Lemma 6.3.5. *Let (M, g) be a Riemannian n-manifold. Let (E_i) be a local frame of TM. Let ∇^g denote the Levi-Cività connection and let $f \in C^\infty(M)$. Then we locally have*

$$\Delta(f) = -\sum_{i=1}^{n} \left(E_i^2(f) - (\nabla^g_{E_i} E_i)(f) \right).$$

Proof. The divergence of a vector field $X \in \mathfrak{X}(M)$ is given in general by

$$\mathrm{div}(X) = \mathrm{tr}^g(\nabla^g X) = \sum_{i=1}^{n} g(\nabla^g_{E_i} X, E_i).$$

Then using lemma 6.3.4 we compute

$$\begin{aligned} \nabla^g_{E_i}(\mathrm{grad}(f)) &= \sum_{j=1}^{n} \nabla^g_{E_i}(E_j(f)E_j) \\ &= \sum_{j=1}^{n} \left(E_i E_j(f) E_j + E_j(f) \nabla^g_{E_i} E_j \right). \end{aligned}$$

So we have

$$\begin{aligned} \Delta(f) &= -\sum_{i,j=1}^{n} \left(g(E_i E_j(f) E_j, E_i) + E_j(f) g(\nabla^g_{E_i} E_j, E_i) \right) \\ &= -\sum_{i,j=1}^{n} E_i E_j(f) \delta_{ji} - \sum_{i,j=1}^{n} E_j(f) g(\nabla^g_{E_i} E_j, E_i) \\ &= -\sum_{i=1}^{n} E_i^2(f) + \sum_{i,j=1}^{n} E_j(f) g(E_j, \nabla^g_{E_i} E_i) \\ &= -\sum_{i=1}^{n} \left(E_i^2(f) - (\nabla^g_{E_i} E_i)(f) \right). \end{aligned}$$

where we used that for an orthonormal frame we have

$$0 = E_i g(E_i, E_j) = g(\nabla^g_{E_i} E_i, E_j) + g(E_i, \nabla^g_{E_i} E_j).$$

□

Lemma 6.3.6. *Let $(E, \langle \cdot, \cdot \rangle, \nabla) \to (M, g)$ be a Hermitian vector bundle with metric connection over a Riemmanian n-manifold. Let (E_i) be a local orthonormal frame of TM and let $\psi \in \Gamma(E)$ be a section. Then we have*

$$\Delta(|\psi|^2) = 2\mathrm{Re}(\langle \nabla^* \nabla \psi, \psi \rangle) - 2 \sum_{i=1}^{n} |\nabla_{E_i} \psi|^2.$$

Here ∇^* denotes the formal adjoint of ∇ as defined in definition 4.1.9.

Proof. We have

$$\begin{aligned} -\sum_{i=1}^{n} E_i^2 \langle \psi, \psi \rangle &= -\sum_{i=1}^{n} E_i \left(\langle \nabla_{E_i} \psi, \psi \rangle + \langle \psi, \nabla_{E_i} \psi \rangle \right) \\ &= -\sum_{i=1}^{n} \left(\langle \nabla_{E_i} \nabla_{E_i} \psi, \psi \rangle + 2 \langle \nabla_{E_i} \psi, \nabla_{E_i} \psi \rangle + \langle \psi, \nabla_{E_i} \nabla_{E_i} \psi \rangle \right) \end{aligned}$$

$$= -2\sum_{i=1}^{n} |\nabla_{E_i}\psi|^2 - 2\sum_{i=1}^{n} \mathrm{Re}(\langle \nabla_{E_i}\nabla_{E_i}\psi, \psi\rangle).$$

On the other hand we have

$$\begin{aligned}\nabla^*\nabla\psi = -\mathrm{tr}^g(\nabla(\nabla\psi)) &= -\sum_{i=1}^{n} \nabla_{E_i}(\nabla\psi)(E_i)\\ &= -\sum_{i=1}^{n}\left(\nabla_{E_i}\nabla_{E_i}\psi - \nabla\psi(\nabla^g_{E_i}E_i)\right)\\ &= -\sum_{i=1}^{n}\left(\nabla_{E_i}\nabla_{E_i}\psi - \nabla_{\nabla^g_{E_i}E_i}\psi\right), \end{aligned} \tag{6.1}$$

where ∇^g is the Levi-Cività connection. So then we have

$$\langle \nabla^*\nabla\psi, \psi\rangle = -\sum_{i=1}^{n}\langle \nabla_{E_i}\nabla_{E_i}\psi, \psi\rangle + \sum_{i=1}^{n}\langle \nabla_{\nabla^g_{E_i}E_i}\psi, \psi\rangle.$$

Hence

$$\begin{aligned}2\mathrm{Re}(\langle \nabla^*\nabla\psi, \psi\rangle) &= -2\sum_{i=1}^{n}\mathrm{Re}(\langle \nabla_{E_i}\nabla_{E_i}\psi, \psi\rangle) + \sum_{i=1}^{n}\left(\langle \nabla_{\nabla^g_{E_i}E_i}\psi, \psi\rangle + \langle \psi, \nabla_{\nabla^g_{E_i}E_i}\psi\rangle\right)\\ &= -2\sum_{i=1}^{n}\mathrm{Re}(\langle \nabla_{E_i}\nabla_{E_i}\psi, \psi\rangle) + \sum_{i=1}^{n}(\nabla^g_{E_i}E_i)(|\psi|^2)\end{aligned}$$

Combining this with equation (6.1), we obtain

$$-\sum_{i=1}^{n} E_i^2\langle \psi, \psi\rangle + \sum_{i=1}^{n}(\nabla^g_{E_i}E_i)(|\psi|^2) = 2\mathrm{Re}(\langle \nabla^*\nabla\psi, \psi\rangle) - 2\sum_{i=1}^{n}|\nabla_{E_i}\psi|^2.$$

So by lemma 6.3.5 we obtain

$$\Delta(|\psi|^2) = 2\mathrm{Re}(\langle \nabla^*\nabla\psi, \psi\rangle) - 2\sum_{i=1}^{n}|\nabla_{E_i}\psi|^2.$$

□

Lemma 6.3.7. *Let M be a compact Riemannian four-manifold. Suppose that (ψ, A) is a solution to the Seiberg-Witten equations. Then for every $x \in M$ we have*

$$|\psi(x)|^2 \leq \max(\max_{y\in M}(4|\eta_+(y)| - \mathrm{scal}(y)), 0).$$

Proof. Note that this inequality is invariant under gauge transformations. Therefore we may use theorem 5.2.1, which says that every solution to the Seiberg-Witten equations is gauge equivalent to a smooth solution, to show this inequality. So assume without loss of generality that (ψ, A) is a smooth solution. Then by the Weitzenböck formula and Seiberg-Witten equations we have

$$0 = \nabla^*\nabla\psi + \frac{\mathrm{scal}}{4}\psi + \frac{|\psi|^2}{4}\psi - i\eta_+\cdot\psi.$$

Let x_0 be a point of M where $|\psi(x)|^2$ attains its maximum. Then taking the inner product with ψ and evaluating at x_0 we obtain

$$\langle \nabla^*\nabla\psi(x_0), \psi(x_0)\rangle + \frac{\mathrm{scal}(x_0)}{4}|\psi(x_0)|^2 + \frac{|\psi(x_0)|^4}{4} + \frac{i}{2}\langle \eta_+(x_0)\cdot\psi(x_0), \psi(x_0)\rangle = 0.$$

Let (E_i) be a local orthonormal frame for TM. Then by lemma 6.3.6

$$\Delta(|\psi(x)|^2) + 2\sum_{i=1}^{4} |\nabla_{E_i}(\psi(x))|^2 = 2\mathrm{Re}(\langle \nabla^*\nabla\psi(x), \psi(x)\rangle).$$

Then in a local maximum x_0 we have $\Delta(|\psi(x_0)|^2) \geq 0$. So $\mathrm{Re}(\langle \nabla^*\nabla\psi(x), \psi(x)\rangle) \geq 0$. Therefore we obtain

$$0 \geq \frac{\mathrm{scal}(x_0)}{4}|\psi(x_0)|^2 + \frac{|\psi(x_0)|^4}{4} + \mathrm{Re}(\frac{i}{2}\langle \eta_+(x_0)\cdot\psi(x_0), \psi(x_0)\rangle).$$

Then the Cauchy-Schwarz inequality gives us a bound on $\mathrm{Re}(\frac{i}{2}\langle \eta_+(x_0)\cdot\psi(x_0), \psi(x_0)\rangle)$, namely at most

$$|\eta_+(x_0)||\psi(x_0)|^2,$$

where we used lemma 6.3.1. So we have $|\psi(x_0)| = 0$, in which case $\psi \equiv 0$, or we obtain the bound

$$0 \geq \frac{\mathrm{scal}(x_0)}{4} + \frac{|\psi(x_0)|^2}{4} - |\eta_+(x_0)|,$$

so

$$|\psi(x_0)|^2 \leq 4|\eta_+(x_0)| - \mathrm{scal}(x_0),$$

and the asserted inequality follows, since for all $x \in M$,

$$|\psi(x)|^2 \leq |\psi(x_0)|^2 \leq 4|\eta_+(x_0)| - \mathrm{scal}(x_0) \leq \max(\max_{y\in M}(4|\eta_+(y)| - \mathrm{scal}(y)), 0)$$

□

Corollary 6.3.8. *Let M be a compact Riemannian four-manifold. Suppose that (ψ, A) is a solution to the Seiberg-Witten equations. Then for every $x \in M$ we have*

$$|F_A^+(x)| \leq \frac{1}{2}\max(\max_{y\in M}(4|\eta_+(y)| - \mathrm{scal}(y)), 0) + |\eta_+(x)|.$$

Proof. The Seiberg-Witten equation gives $F_A^+ = \overline{\psi}\otimes\psi - \frac{|\psi|^2}{2}\mathrm{id} - i\eta_+$, so $|F_A^+(x)| \leq \frac{1}{2}|\psi(x)|^2 + |\eta_+(x)|$, so the result follows from the previous one. □

We need the following lemma to obtain our first major result about compactness.

Lemma 6.3.9. *Let (M, g) be a Riemannian manifold of dimension 4. Let $\omega \in \Omega^2(M)$. We may write $\omega = \omega^+ + \omega^-$ where ω^+ is self-dual and ω^- is anti-self-dual. Then we have*

$$\int_M \omega^2 = \|\omega^+\|_{L^2}^2 - \|\omega^-\|_{L^2}^2.$$

Proof. We have

$$\omega^\pm \wedge \omega^\pm = \pm\omega^\pm \wedge \star\omega^\pm = \pm\langle \omega^\pm, \omega^\pm\rangle \mathrm{vol}_g = \pm\|\omega^\pm\|^2\mathrm{vol}_g,$$

and

$$\begin{aligned}\langle \omega^+, \omega^-\rangle\mathrm{vol}_g = \langle \omega^-, \omega^+\rangle\mathrm{vol}_g = \omega^- \wedge \star\omega^+ = \omega^- \wedge \omega^+ \\ = \omega^+ \wedge \omega^- = -\omega^+ \wedge \star\omega^- = -\langle \omega^+, \omega^-\rangle\mathrm{vol}_g,\end{aligned}$$

so $\omega^{\pm} \wedge \omega^{\mp} = 0$. So we have

$$\begin{aligned}\int_M \omega^2 &= \int_M (\omega^+ + \omega^-)^2 \\ &= \int_M (\|\omega^+\|^2 - \|\omega^-\|^2)\mathrm{vol}_g \\ &= \left(\|\omega^+\|_{L^2}^2 - \|\omega^-\|_{L^2}^2\right).\end{aligned}$$

□

In the next theorem we use the following notation:

$$\begin{aligned}\kappa_M^- &= \max_{x\in M}(0, -\mathrm{scal}(x)) \\ \kappa_{M,\eta}^- &= \max(\max_{y\in M}(4|\eta_+(y)| - \mathrm{scal}(y)), 0).\end{aligned}$$

Theorem 6.3.10. *Let M be a compact Riemannian four-manifold. Then there are only finitely many spin^c structures up to isomorphism for M such that the moduli space of solutions to the Seiberg-Witten equations is non-empty and has non-negative formal dimension. For any solution (ψ, A) to the Seiberg-Witten equations at which the formal dimension is non-negative and for any $x \in M$ we have*

$$\begin{aligned}|\psi(x)|^2 &\leq \kappa_{M,\eta}^- \\ \|\nabla(\psi)\|_{L^2}^2 &\leq \left(\frac{\kappa_M^-}{4} + \frac{1}{2}\kappa_{M,\eta}^- + \|\eta_+\|_\infty\right)\kappa_{M,\eta}^-\mathrm{vol}(M) \\ |F_A^+(x)| &\leq \frac{1}{2}\kappa_{M,\eta}^- + |\eta_+(x)| \\ \|F_A^+\|_{L^2}^2 &\leq \frac{(\kappa_{M,\eta}^- + 2|\eta_+(x)|)^2}{4}\mathrm{vol}(M) \\ \|F_A^-\|_{L^2}^2 &\leq \frac{(\kappa_{M,\eta}^- + 2|\eta_+(x)|)^2}{4}\mathrm{vol}(M) - 8\pi^2\chi(M) - 12\pi^2\sigma(M).\end{aligned}$$

Proof. The first and third inequality were already established above. The fourth one follows from the third one by integrating over M. For the fifth one, we have formal dimension

$$c_1(\mathcal{L})^2 - (2\chi(M) + 3\sigma(M)) \geq 0$$

by corollary 5.2.5 and by lemma 6.3.9 we have

$$c_1(\mathcal{L})^2 = \frac{1}{4\pi^2}\left(\|F_A^+\|_{L^2}^2 - \|F_A^-\|_{L^2}^2\right),$$

Combining these two facts yields the fifth inequality. Finally, the second inequality we obtain from the Weitzenbock formula

$$0 = \nabla^*\nabla\psi + \frac{\mathrm{scal}}{4}\psi + \frac{F_A^+}{2}\cdot\psi$$

and the bound on $|F_A^+(x)|$ for all $x \in M$.

It remains to show that there are only finitely many spin^c structures up to isomorphism with non-empty moduli space and non-negative formal dimension. Suppose we have a solution (ψ, A) of the Seiberg-Witten equations at which the formal dimension is non-negative. Then by the preceeding, we have a bound depending only on the geometry of M and the perturbation parameter η on both $\|F_A^+\|_{L^2}^2$ and $\|F_A^-\|_{L^2}^2$. So the cohomology class represented by $\frac{i}{2\pi}F_A$ lies in a compact subset of $H^2(M,\mathbb{R})$. Moreover, since this is the Chern class it must also be integral, so there are only finitely many possibilities for this class. So there are only finitely many spin^c structures whose determinant line bundle has this given first Chern class. □

6.4 Compactness

To show compactness we will first show that we can always find a representative of a point in the moduli space subject to specific conditions. This process is called gauge fixing. We can then use this fixed gauge to obtain more a priori bounds for solutions of the Seiberg-Witten equations. Using this bounds for low regularity solutions, we can use elliptic bootstrapping to obtain bounds for higher regularity solutions as well. Finally, we will use all these bounds to argue that the moduli space is compact.

6.4.1 Gauge fixing

The goal of this section is to fix gauge and find a specific representative for a solution to the Seiberg-Witten equations.

Lemma 6.4.1 (Gauge-fixing Lemma). *Let $\mathcal{L}$ be a complex line bundle over a compact Riemannian four-manifold M with a hermitian metric. Fix a unitary C^∞ connection A_0 on $\mathcal{L}$. Then for any $\ell \geq 0$ there are constants $K, C > 0$ depending only on M, A_0, ℓ such that the following hold: For any $L^{\ell,2}$ unitary connection A on $\mathcal{L}$ there is an $L^{\ell+1,2}$ change of gauge σ such that $\sigma \cdot A = A_0 + \alpha$ where $\alpha \in L^{\ell,2}(T^*M \otimes i\mathbb{R})$ satisfies $d^*\alpha = 0$ and*

$$\|\alpha\|_{L^{\ell,2}}^2 \leq C\|F_A^+\|_{L^{\ell-1,2}}^2 + K.$$

Proof. Let $a_0 = A - A_0 \in L^{\ell,2}(T^*M \otimes i\mathbb{R})$. By the Hodge decomposition we find $f \in L^{\ell+1,2}(M, i\mathbb{R})$, $\beta \in L^{\ell+1,2}(\wedge^2 T^*M \otimes i\mathbb{R})$ and ω a harmonic 1-form such that

$$a_0 = df + d^*\beta + \omega.$$

Then $\gamma := \exp\left(\frac{1}{2}f\right) \in L^{\ell+1,2}$. Then we have $d\gamma = \frac{1}{2}\gamma df$ and we have

$$\gamma \cdot A = A - 2\gamma^{-1}d\gamma = A - df = A_0 + df + d^*\beta + \omega - df = A_0 + d^*\beta + \omega.$$

Define $\alpha = d^*\beta + \omega$. Now, F_A^+ is the self-dual part of the curvature of the connection A, so in particular we have $d^+A = F_A^+$. Then it follows that

$$(d^* + d^+)(d^*\beta + \omega) = d^+(d^*\beta + \omega) = d^+(\gamma \cdot A - A_0) = F_A^+ - F_{A_0}^+.$$

Then ellipticity of (d^*, d^+) gives us the following bound for a constant C depending only on M and ℓ (here and in the following C may be increased between estimates)

$$\|d^*\beta\|_{L^{\ell,2}}^2 \leq C\|(d^*d^*\beta, d^+d^*\beta)\|^2 = C\|F_A^+ - F_{A_0}^+\|_{L^{\ell-1,2}}^2 \leq C\|F_A^+\|_{L^{\ell-1,2}}^2 + C\|F_{A_0}^+\|_{L^{\ell-1,2}}^2.$$

Then with $K_1 = \|F_{A_0}^+\|_{L^{\ell-1,2}}^2$ we have

$$\|d^*\beta\|_{L^{\ell,2}}^2 \leq C\|F_A^+\|_{L^{\ell-1,2}}^2 + CK_1.$$

Next, we need to bound the harmonic component ω. We do this by applying a further gauge transformation.

Let H^1 denote the space of purely imaginary harmonic 1-forms. Let $\omega_0 \in H^1$ and assume that all periods of ω_0 lie in $2\pi i\mathbb{Z}$. Let $\widetilde{M}$ be the universal cover of M. Then integrating ω_0 along curves starting at a base point $x_0 \in M$ gives a map $f\colon \widetilde{M} \to i\mathbb{R}$. Then $\widetilde{\varphi} = \exp(f)$ descends to a map $\varphi\colon M \to S^1$ since all periods lie in $2\pi i\mathbb{Z}$. Then $d\varphi = \varphi df = \varphi\omega_0$, so $\omega_0 = \varphi^{-1}d\varphi$.

Let Λ denote the lattice of purely imaginary harmonic 1-forms in H^1 with periods in $4\pi i\mathbb{Z}$. Then H^1/Λ is a torus, so there exists a constant K_2 depending on ℓ such that any ω_0 can be written as $\omega_0 = \omega_1 + 2\omega_2$ with $\omega_2 \in \frac{1}{2}\Lambda$ and $\|\omega_1\|^2_{L^{\ell,2}} \leq K_2$.

We now apply this to our harmonic component $\omega = \omega_1 + 2\omega_2$. Then ω_2 has periods in $2\pi i\mathbb{Z}$, so there exists a $\varphi\colon M \to S^1$ such that $\omega_2 = \varphi^{-1}d\varphi$. So we have

$$\varphi \cdot (A_0 + d^*\beta + \omega) = A_0 + d^*\beta + \omega - 2\varphi^{-1}d\varphi = A_0 + d^*\beta + \omega_1.$$

So we for $\alpha = d^*\beta + \omega_1$ and $\sigma = \varphi\gamma$ we have $d^*\alpha = 0$, $\sigma \cdot A = A_0 + \alpha$ and

$$\|\alpha\|^2_{L^{\ell,2}} \leq \|\omega_1\|^2_{L^{\ell,2}} + \|d^*\beta\|^2_{L^{\ell,2}} \leq K_2 + CK_1 + C\|F_A^+\|^2_{L^{\ell-1,2}},$$

so with $K = K_2 + CK_1$ the result follows. □

6.4.2 More bounds

In section 6.3 we found bounds on the spinor ψ and the curvature F_A^+ which essentially only depended on the geometry of M. In this section we will obtain bounds on dF_A^+ and A. To do this, we need the following lemma.

Lemma 6.4.2. *Let M be a manifold and let ∇ be a torsion-free connection on TM. Let* Alt *denote the antisymmetrization map. Then we have for $\omega \in \Omega^k(M)$*

$$d\omega = \mathrm{Alt}(\nabla\omega),$$

(where ∇ is the induced connection on $\wedge^k T^*M$).

Proof. It suffices to check this on 1-forms, since both sides are anti-derivations. So let $\omega \in \Omega^1(M)$ and $X, Y \in \mathfrak{X}(M)$. Then we have

$$\begin{aligned}\mathrm{Alt}(\nabla\omega)(X,Y) &= (\nabla_X\omega)(Y) - (\nabla_Y\omega)(X)\\ &= X(\omega(Y)) - \omega(\nabla_X Y) - Y(\omega(X)) + \omega(\nabla_Y X)\\ &= X(\omega(Y)) - Y(\omega(X)) - \omega(\nabla_X Y - \nabla_Y X)\\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = d\omega(X,Y).\end{aligned}$$

□

Then we have the following result for the exterior derivative of F_A^+.

Lemma 6.4.3. *There is a constant C depending only on M and the perturbation parameter η such that for any solution (ψ, A) to the Seiberg-Witten equations we have*

$$\|dF_A^+\|^2_{L^2} \leq C.$$

Proof. Let ∇^g denote the Levi-Cevita connection. Then we have by the spinor connection property and the Seiberg-Witten equation that

$$\nabla^g F_A^+ = \nabla(\overline{\psi} \otimes \psi - \frac{|\psi|^2}{2}\mathrm{id}) - i\nabla^g \eta_+.$$

By lemma 6.4.2, the anti-symmetrization of the Levi-Civita connection applied to a 2-form is precisely the exterior derivative of the 2-form, so by antisymmetrizing the equation above we obtain

$$dF_A^+ = \nabla(\overline{\psi} \otimes \psi - \frac{|\psi|^2}{2}) - id\eta_+.$$

Then we obtain

$$dF_A^+ = \nabla(\overline{\psi}) \otimes \psi + \overline{\psi} \otimes \psi - \mathrm{Re}\langle \nabla\psi, \psi\rangle \mathrm{id} - id\eta_+.$$

All terms on the right hand side are L^2-bounded with bounds only depending on the geometry of M and the perturbation parameter, so we obtain a bound C for $\|dF_A\|_{L^2}^2$ □

Lemma 6.4.4. *There is a constant C_1 only depending on M and the perturbation parameter η such that for any solution (ψ, A) to the Seiberg-Witten equations we have*

$$\|F_A^+\|_{L^{1,2}}^2 \leq C_1.$$

Proof. Let $\pi \colon L^{1,2}(\wedge_+^2 T^*M) \to H^\perp$ be the orthogonal projection to the orthogonal complement of the space of harmonic self-dual two forms. We may then write $F_A^+ = \pi(F_A^+) + \omega$, with ω self-dual and harmonic. Then Hodge theory gives a constant C' such that

$$\|\pi(F_A^+)\|_{L^{1,2}}^2 \leq C' \|dF_A^+\|_{L^2}^2.$$

We also have an orthogonal projection to self-dual harmonic two forms, so there is a constant $C'' > 0$ such that $\|\omega\|_{L^{1,2}} \leq C''\|F_A^+\|_{L^2}$. Since we have an L^2-bound on F_A^+ and a L^2 bound on dF_A^+ it follows that the asserted $C_1 > 0$ exists. □

Then we can combine these bounds with gauge fixing to obtain the following statement.

Proposition 6.4.5. *Let σ be a spin^c structure and let A_0 be a fixed C^∞ connection on the determinant bundle* $\det(\sigma)$*. Then there exists a constant K_1 depending only on M, the perturbation parameter η and A_0 such that for any solution (ψ, A) to the Seiberg-Witten equations we have a connection $A' = A_0 + \alpha$ gauge equivalent to A with $d^*\alpha = 0$ and $\|\alpha\|_{L^{2,2}}^2 \leq K_1$.*

Proof. This follows from lemma 6.4.1 combined with the previous lemma. □

6.4.3 (Sequential) compactness of the moduli space

The following theorem is the key to showing compactness of the moduli space.

Theorem 6.4.6. *Suppose that (ψ, A) is a solution to the Seiberg-Witten equations and that we have fixed gauge so that $A = A_0 + \alpha$ where A_0 is a fixed C^∞ connection on the determinant line bundle, with $d^*\alpha = 0$ and with the projection of α into the harmonic forms contained in a given compact fundamental domain modulo the lattice of harmonic forms with periods in $4\pi i\mathbb{Z}$. For every $\ell \geq 2$ there is a constant $C(\ell)$, depending only on M, A_0, the perturbation parameter and ℓ such that*

$$\|\alpha\|_{L^{\ell,2}}^2 + \|\psi\|_{L^{\ell,2}}^2 \leq C(\ell).$$

(Here the $L^{\ell,2}$-norm of the spinor is taken with respect to ∇_{A_0}).

Proof. We have proven that ψ is pointwise bounded and we have seen that α is $L^{2,2}$ bounded. We have $\nabla_A \psi = \nabla_{A_0}\psi + \alpha\psi$, so the $L^{2,2}$-bound on ψ together with the L^2 bound on ∇_A yield a $L^{1,2}$ bound on ψ.

We will show that ψ is bounded in $L^{3,2}$. By the Dirac equation we have

$$\not{D}_{A_0}\psi = -\alpha \cdot \psi. \tag{6.2}$$

Since α is $L^{2,2}$ bounded and ψ is C^0 bounded, it follows that $\not{D}_{A_0}\psi$ is L^4 bounded, so by ellipticity of $\not{D}_{A_0}$, the projection to the orthogonal of $\ker(\not{D}_{A_0})$ is $L^{1,4}$ bounded. Since ψ is also L^2 bounded, so is its projection to $\ker(\not{D}_{A_0})$. Since this is finite dimensional, all norms are equivalent, and so the projection is also $L^{1,4}$ bounded. So together these imply a $L^{1,4}$ bound on ψ.

Then using lemma 6.1.4, we obtain an $L^{1,3}$ bound on $\not{D}_{A_0}\psi$, so arguing in the same way, we obtain a $L^{2,3}$ bound on ψ. Then using lemma 6.1.3, we obtain a $L^{2,2}$ bound on $\not{D}_{A_0}\psi$ and so once again we obtain a $L^{3,2}$ bound on ψ.

Now, from the curvature equation

$$F_A^+ = \overline{\psi} \otimes \psi - \frac{|\psi|^2}{2}\mathrm{id} - i\eta_+ \tag{6.3}$$

and lemma 6.1.2 it follows that F_A^+ is also bouded in $L^{3,2}$. So by the gauge fixing lemma 6.4.1 α is $L^{4,2}$ bounded. This was the initial step of obtaining a bound $C(\ell)$.

Now suppose by induction that we have for some $\ell \geq 3$ bounds for the $L^{\ell,2}$-norms of α and ψ. Then from equation 6.2 and lemma 6.1.2 it follows that there is a $L^{\ell,2}$ bound on $\not{D}_{A_0}\psi$ and so there is a $L^{\ell+1,2}$ bound on ψ. Then from the curvature equation 6.3 it follows that there is a $L^{\ell,2}$-bound on F_A^+. So by the gauge fixing lemma 6.4.1, there is a $L^{\ell+1,2}$ bound on α. The result follows by induction. □

From this theorem we obtain sequential compactness of the moduli space.

Corollary 6.4.7. *Let (ψ_n, A_n) be any sequence of solutions to the Seiberg-Witten equations. Then after passing to a subsequence, and applying $L^{3,2}$ gauge transformations we can arrange that the (ψ_n, A_n) are C^∞ objects and they converge in the C^∞ topology to a limit (ψ, A) which is also a solution to the Seiberg-Witten equations.*

Proof. By Morrey's theorem, theorem 4.2.9, we have compact embeddings $L^{\ell,2} \hookrightarrow C^{\ell-3}$. By the theorem we can gauge fix each (ψ_n, A_n) with a $L^{3,2}$ gauge transformation to obtain a sequence of $(\psi_n, A_0 + \alpha_n)$ as in the theorem. For these α_n and ψ_n we have $L^{\ell,2}$ bounds only depending on ℓ, M, a choice of A_0 and the perturbation parameter η. We now apply a diagonal argument to obtain a C^∞ convergent subsequence.

We inductively define subsequences $(\psi_n^{(\ell)}, A_0 + \alpha_n^{(\ell)})$ for all $\ell \geq 3$. For $\ell = 3$, we have a compact embedding $(\psi_n, A_0 + \alpha_n) \in C^0$. Since we have an a priori $L^{3,2}$ bound on ψ_n and α_n, we obtain a convergent subsequence $(\psi_n^{(3)}, A_0 + \alpha_n^{(3)})$. Now, suppose we have defined $(\psi_n^{(\ell)}, A_0 + \alpha_n^{(\ell)})$. Then by the compact embedding $L^{\ell+1,2} \hookrightarrow C^{\ell-2}$ and the $L^{\ell+1,2}$ bound on the ψ_n and α_n, we have a convergent subsequence $(\psi_n^{(\ell+1)}, A_0 + \alpha_n^{(\ell+1)})$ of $(\psi_n^{(\ell)}, A_0 + \alpha_n^{(\ell)})$.

Now, we define $(\mu_n, B_n) = (\psi_n^{(n+3)}, A_0 + \alpha_n^{(n+3)})$. Then we have that (μ_n, B_n) is a subsequence of (ψ_n, A_n) and so it is convergent for all C^k. So it is a C^∞ convergent sequence, say $(\mu_n, B_n) \to (\mu, B)$. Since all (μ_n, B_n) solve the Seiberg-Witten equations, it follows that (μ, B) is also a solution to the Seiberg-Witten equations. □

Corollary 6.4.8 (Compactness of the moduli space)**.** *The moduli space* $\mathscr{M}_\eta^{k+1}$ *is compact.*

Proof. Since Sobolev spaces are seperable, they are second countable. Therefore the notions of compactness and sequential compactness coincide. The previous corollary precisely states that the moduli space is sequentially compact. □

Lecture 7

—

Lecture 8

Seiberg-Witten equations on cylinders

8.1 Applications

Before we actually start to study the Seiberg-Witten equations on cylinders, we will give the proof for two theorems that require these kind of objects, where we black box the results from the analysis of Seiberg-Witten equations on cylinders.

8.1.1 Vanishing for connected sums

Theorem 8.1.1. *Let M_1, M_2 be closed four-manifolds with $b_2^+(M_1), b_2^+(M_2) > 0$. Then $M_1 \# M_2$ has vanishing Seiberg-Witten invariants*

Proof. Let $D_1 \subseteq M_1$ and $D_2 \subseteq M_2$ be open discs, let $N_i := (M_i \setminus D_i) \cup_{S^3} S^3 \times [0, \infty)$ and smooth out the corner, such that N_i gets a cylindrical end diffeomorphic to $S^3 \times [1, \infty)$. Equip N_i with a generic metric g_i such that $g_i|_{S^3 \times [1,\infty)} \cong g_{S^3} + dt^2$. For every $r > 1$, pick a diffeomorphism $M_1 \# M_2 \cong (N_1 \setminus (S^3 \times (r, \infty))) \cup_{S^3 \times \{r\}} (N_2 \setminus (S^3 \times (r, \infty)))$. This process gives a family of metrics g_r on $M_1 \# M_2$ such that g_r is generic away from the cylindrical neck. Also pick a $Spin^c$-structure σ on $M_1 \# M_2$ and equip M_1 and M_2 with the respective $Spin^c$-structure σ_1 and σ_2, which are canonically induced from $\sigma|_{M_i \setminus D_i}$.

We will use the following black box: for $r \gg 1$ and generic η with $\mathrm{supp}(\eta) \subseteq M_1 \setminus D_1 \sqcup M_2 \setminus D_2$,

$$\mathscr{M}_{g_r,\eta}(\sigma, *) \cong \mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1, \sigma_1, *) \times \mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2, \sigma_2, *) .$$

Here, $\mathscr{M}_{g_i,\eta|_{N_i},\mu}(N_i, \sigma_i, *)$ is the moduli space of solutions to $SW_{\eta|_{N_i}}$ with suitable exponential convergence to a suitable model at ∞, modulo elements of the gauge group that are 1 at a chosen basepoint. The residual S^1 action on $\mathscr{M}_{g_r,\eta}(\sigma, *)$ acts diagonally on the right hand side. Let $\mathscr{M}_{g_i,\eta|_{N_i},\mu}(N_i, \sigma_i) := \mathscr{M}_{g_i\eta|_{N_i},\mu}(N_i, \sigma_i, *)/S^1$. Since we assumed M_1, M_2 had positive b_2^+, the same is true for N_1, N_2, so a black box tells us the associated moduli spaces generically consist of irreducible solutions. We see that, generically,

$$\dim(\mathscr{M}_{g_r,\eta}(\sigma)) = \dim(\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1, \sigma_1)) + \dim(\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2, \sigma_2)) + 1 .$$

In particular, if $\dim(\mathscr{M}_{g_r,\eta}(\sigma)) = 0$, one of the moduli spaces $\mathscr{M}_{g_i,\eta|_{N_i},\mu}(N_i, \sigma_i)$ must be generically empty, such that $\mathscr{M}_{g_r,\eta}(\sigma)$ is also generically empty for r big enough.

In the other case, let θ_0 be a global angular form for the residual S^1 action on $\mathscr{M}_{g_r,\eta}(\sigma,*)$, and let θ_1, θ_2 be global angular forms for the residual S^1-actions on $\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*)$ and $\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*)$. In particular, we see $\theta_0 = \frac{1}{2}(\mathrm{pr}_1^*\theta_1 + \mathrm{pr}_2^*\theta_2) + \text{exact term}$. We see

$$\begin{aligned}\mathrm{sw}(s) &= \int_{\mathscr{M}_{g_r,\eta}(\sigma,*)} (\theta_0 \wedge (d\theta_0)^n) \\ &= \tfrac{1}{2^{n+1}} \int_{\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*)\times\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*)} (\mathrm{pr}_1^*\theta_1 + \mathrm{pr}_2^*\theta_2) \wedge (\mathrm{pr}_1^* d\theta_1 + \mathrm{pr}_2^* d\theta_2)^n ,\end{aligned}$$

where $n = \frac{1}{2}\dim(\mathscr{M}_{g_r,\eta}(\sigma))$. Thus,

$$\mathrm{sw}(\sigma) = 2^{-n-1} \sum_k \binom{n}{k} \left(\int_{\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*)} \theta_1 \wedge (d\theta_1)^k \int_{\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*)} (d\theta_2)^{n-k} + 1 \leftrightarrow 2 \right) ,$$

where $1 \leftrightarrow 2$ indicates the same term but with 1 and 2 exchanged. Since $(d\theta)^{n-k}$ is exact whenever $n \neq k$, we find

$$\mathrm{sw}(\sigma) = 2^{-n-1} \left(\int_{\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*)} \theta_1 \wedge (d\theta_1)^n + \int_{\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*)} \theta_2 \wedge (d\theta_2)^n \right) .$$

Since $2n+1 = \dim(\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*)) + \dim(\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*))$, we see that in order to have $\mathrm{sw}(\sigma) \neq 0$, we must have that either $2n+1 = \dim(\mathscr{M}_{g_1,\eta|_{N_1},\mu}(N_1,\sigma_1,*))$ or $2n+1 = \dim(\mathscr{M}_{g_2,\eta|_{N_2},\mu}(N_2,\sigma_2,*))$. But then we have $\dim(\mathscr{M}_{g_i,\eta|_{N_i},\mu}(N_i,\sigma_i,*)) = 0$ for the other one, implying $\dim(\mathscr{M}_{g_i,\eta|_{N_i},\mu}(N_i,\sigma_i)) = -1$, i.e. it's generically empty. So in that case, we generically have $\mathscr{M}_{g_r,\eta}(\sigma) = \emptyset$ as well, so $\mathrm{sw}(\sigma) = 0$. □

If we combine this with Taubes' non-vanishing result, we obtain

Corollary 8.1.2. *Let M be a closed four-manifold admitting symplectic structures. Then M is irreducible: it cannot be decomposed as $M = N_1 \# N_2$ with N_1, N_2 closed four-manifolds satisfying $b_2^+(N_1), b_2^+(N_2) > 0$. In particular, a connected sum of two closed symplectic four-manifolds is never symplectic.*

8.1.2 Thom conjecture

One invariant of four manifolds that feels rather untouchable is the **minimal genus function**:

Definition 8.1.3 (Minimal genus function)**.** Let M be a closed four-manifold. The **minimal genus function** $g_M : H_2(M;\mathbb{Z}) \setminus 0 \to \mathbb{N}_0$ sends a homology class α to the minimal genus of a closed embedded surface $\Sigma \subseteq M$ representing α.

While this is a rather strong invariant of four manifolds, it is not very computable. One can make educated guesses, for instance, an application of the adjunction formula and Riemann-Roch for curves gives

Theorem 8.1.4. *Let $M = \mathbb{C}P^2$ with the standard complex structure and standard generator H of $H_2(M;\mathbb{Z})$. A holomorphic curve Σ in M representing a class dH with $d > 0$ satisfies*

$$\mathrm{genus}(\Sigma) = \frac{(d-1)(d-2)}{2} .$$

If $d < 0$, we can represent classes in dH using antiholomorphic curves (i.e. holomorphic curves, but with the opposite orientation), such that we are lead to the following conjecture attributed to Thom, which was proven by Kronheimer and Mrowka using Donaldson theory, but for which we will sketch a proof using Seiberg-Witten theory:

Theorem 8.1.5 (Thom conjecture). *The minimal genus function $g_{\mathbb{C}P^2}$ of $\mathbb{C}P^2$ is*

$$g_{\mathbb{C}P^2}(dH) = \max(0, \frac{(|d|-1)(|d|-2)}{2}). \tag{8.1}$$

Proof. The case where $|d| = 0,1,2,3$ follow from a classical result by Kervaire and Milnor that $g_{\mathbb{C}P^2}(|d|) \geq 1$ whenever $|d| > 2$, so we will do the $|d| > 3$ case. Moreover, we only have to prove it when $d \geq 0$, since that also implies the result for $d < 0$, so we may assume $d > 3$.

Suppose Σ is a genus g surface representing dH. Then Σ is genus minimising if and only if the proper transform $\widetilde{\Sigma}$ of Σ in $\mathbb{C}P^2 \# d^2\overline{\mathbb{C}P^2}$ is genus minimising in $dH - \sum_i E_i$. Since we blew up all self-intersections of Σ, we see that $\widetilde{\Sigma} \cdot \widetilde{\Sigma} = 0$, so we can find a tubular neighbourhood $U \cong \Sigma \times D^2$ of $\widetilde{\Sigma}$. Write $N := \partial U \cong \Sigma \times S^1$. We can equip Σ with a constant curvature metric g_0, such that the curvature s_0 satisfies

$$\mathrm{vol}_{g_0}(\Sigma) s_0 = 4\pi(2 - 2g(\Sigma))$$

by the Gauss-Bonnet theorem and we equip N with the product metric $g_N = g_0 + d\theta^2$.

We can equip $M := \mathbb{C}P^2 \# d^2\overline{\mathbb{C}P^2}$ with a generic metric g_1 such that N has a tubular neighbourhood U_N isometric to $N \times [-1,1]$ with the product metric $g_n + dt^2$. We can "stretch the neck" to find a family of metrics g_n such that $g_n|_{M \setminus U_N} = g_m|_{M \setminus U_N}$ and $(U_N, g_n|_{U_N}) \cong (N \times [-n,n], g_N + dt^2)$.

Black box: if we pick n large enough, then $(g_n, 0)$ will lie in the positive chamber corresponding to the canonical *Spin*^{c} structure σ_0 corresponding to $-K = 3H - \sum_i E_i$. Moreover, $sw^+(\sigma_0) = 1$ since M is a Kähler surface, so we conclude that there are $(\sigma_0, g_n, 0)$-monopoles whenever n is big enough, so we can pick a monopole (A_n, ψ_n) for each g_n. Since the scalar curvature of g_n is uniformly bounded, the key estimate implies that $\|\psi_n\|_\infty < C$ for some constant C independent of n.

Moreover, recall that solutions to the unperturbed Seiberg-Witten equations on a compact manifold are stationary points of the energy functional

$$\mathscr{E}(A,\psi) = \int_M (|\nabla_A \psi|^2 + |F_A^+|^2 + \tfrac{s}{4}|\psi|^2 + \tfrac{1}{8}|\psi|^4)\mathrm{vol},$$

such that solutions to the unperturbed Seiberg-Witten equations corresponding to a *Spin*^{c}-structure σ have energy

$$E(\sigma) = -4\pi^2 \int_M c_1^2(\det \sigma).$$

Thus, (A_n, ψ_n) has energy $4\pi^2(d^2 - 9)$ on M. Moreover, $\mathscr{E}(\psi_n|_{M\setminus U_N}, A_n|_{M \setminus U_N})$ is bounded from below by $\int_{M \setminus U_N} \frac{s}{4}|\psi_n|^2$, so because s is uniformly bounded and $|\psi_n|$ is uniformly bounded, we conclude that

$$\mathscr{E}(A_n|_{U_N}, \psi_n|_{U_N}) = E(\sigma_0) - \mathscr{E}(A_n|_{M\setminus U_N}, \psi_n|_{M \setminus U_N}) < C$$

for some constant C independent of n.

In total, we have found a sequence of monopoles $(A_n, \psi_n)_n$ on $N \times [-n,n]$ with uniformly bounded energy and uniformly bounded $\|\psi_n\|_\infty$. If we pick an isomorphism $\det(\sigma_0)|_{N \times [-n,n]} \cong$

$[-n, n] \times \det(\sigma_0)|_N$ and a reference connection $\hat{\nabla}$ such that on $\det(\sigma_0)|_{N\times[-n,n]}$, $\hat{\nabla} = \nabla_N + dt$, with ∇_N a connection on $\det(\sigma_0)|_N$, we can always pick a gauge such that $A_n = ia_n(t)$, where $a_n(t)$ is a real one-form on $N \times \{t\}$, such that in this gauge, the pair (A_n, ψ_n) defines a one-parameter family of structures living on N.

Black box: since we are stretching the neck to infinity, we will actually find that this defines a monopole (A, ψ) on the three-manifold N corresponding to the $Spin^c$ structure $\sigma_0|_N$, which has spinor bundle $S^+|_N$, where we note that $Spin^c(3) \cong U(2)$, so the fundamental representation is on $\mathbb{C}^2$. For such objects, there is another curvature estimate that states

$$\|\psi\|_\infty^2 \leq -2 \max s \,,$$

so because the scalar curvature on Σ is constant, we conclude

$$\mathrm{vol}_{g_0}(\Sigma)\|\psi\|_\infty^2 \leq 8\pi(2g(\Sigma) - 2)\,.$$

Black box: by analysis of the monopole equations on three-manifolds, we then conclude

$$\mathrm{vol}_{g_0}(\Sigma)\|F_A\|_\infty \leq 2\pi(2g(\Sigma) - 2)\,.$$

Now, if we let $p : N \to \Sigma$ denote the projection, we see

$$c_1(S^+|_N) = p^* c_1(S^+|_\Sigma) = p^*((dH - \sum_i E_i) \cap (3H - \sum_i E_i)) = -p^*(d(d-3))\,,$$

so in particular, since $d > 3$,

$$d(d-3) = \left| \int_\Sigma c_1(S^+|_\Sigma) \right| \leq \tfrac{1}{2\pi} \int_\Sigma |F_A|^2 \leq 2g(\Sigma) - 2\,.$$

In total, we see $g(\Sigma) \geq (d(d-3)+2)/2$, so

$$g(\Sigma) \geq \frac{(d-1)(d-2)}{2}\,. \tag{8.2}$$

□

In fact, similar techniques can be used to prove

Theorem 8.1.6 (Adjunction inequality). *Let M be a closed four-manifold with $b_2^+(M) > 1$ and let $c \in H^2(M; \mathbb{Z})$ be a Seiberg-Witten basic class of M, i.e. $\mathrm{sw}(\sigma_c) \neq 0$. Let $\Sigma \subseteq M$ be a closed embedded surface of genus $g \geq 1$, such that $\Sigma \cdot \Sigma \geq 0$, then*

$$2g - 2 \geq |c \cap [\Sigma]| + \Sigma \cdot \Sigma\,. \tag{8.3}$$

We conclude that if M is a symplectic four-manifold with $b_2^+ > 1$, then symplectic surfaces $\Sigma \subseteq M$ with $\Sigma \cdot \Sigma \geq 0$ are genus minimising in their homology class. In fact, Morgan, Szabó and Taubes have shown that symplectic surfaces are also genus minimising whenever $b_2^+ = 1$ (also using cylindrical gluing techniques) and Osváth and Szabó have proven that this also holds for negative self-intersecting curves (using different techniques):

Theorem 8.1.7 (Symplectic Thom conjecture). *Let M be a closed symplectic four-manifold and let $\Sigma \subseteq M$ be a closed symplectic surface. Then $g_M([\Sigma]) = \mathrm{genus}(\Sigma)$, i.e., Σ is genus-minimising in its homology class.*

8.2 Seiberg-Witten on cylinders

8.2.1 Monopole-equations on 3-manifolds

In this section, we will study $Spin^c$-structures on three-manifolds. Recall that any closed oriented three-manifold M is parallelisable, such that (after picking a metric), $Fr(TM) \cong M \times SO(3)$. Therefore, any closed orientable three-manifold admits a $Spin$-structure corresponding to the double cover $M \times Spin(3) \to M \times SO(3)$, with spinor bundle $M \times \mathbb{C}^2$. Using that $Spin(3) \cong SU(2)$, and the fact that $SU(2) \times_{\mathbb{Z}_2} U(1) = \{e^{i\theta}A | e^{i\theta} \in U(1), A \in SU(2)\}$, we see that $Spin^c(3) \cong U(2)$. The fundamental representation of $Spin^c(3)$ is therefore on $\mathbb{C}^2$.

Proposition 8.2.1. *Let M be a closed oriented three-manifold. Then $Spin^c$-structures σ on M are in one-to-one correspondence with complex line bundles $L \to M$. The spinor bundle associated to σ is $S \cong (M \times \mathbb{C}^2) \otimes L$.*

Given a $Spin^c$-structure σ on M with spinor bundle S, let $\mathscr{A}(S)$ denote the space of spinorial connections. We define the configuration space

$$\mathscr{C}(M, \sigma) \cong \mathscr{A}(S) \times \Gamma(S), \tag{8.4}$$

which, under a choice of reference connection B_0, is isomorphic to $i\Omega^1(M) \times \Gamma(S)$.

Definition 8.2.2 (Chern-Simons-Dirac functional)**.** Let M be a closed oriented three-manifold with $Spin^c$ structure σ with Spinor bundle S and fix a reference connection B_0 on S. The **Chern-Simons-Dirac functional** is the map $\mathscr{L} : \mathscr{C}(M, \sigma) \to \mathbb{R}$ given by

$$\mathscr{L}(B, \phi) = \frac{1}{8} \int_M \mathrm{Tr}(B - B_0) \wedge \mathrm{Tr}(F_B + F_{B_0}) + \frac{1}{2} \int_M \langle \not{D}_B \phi, \phi \rangle \mathrm{vol}\,. \tag{8.5}$$

Proposition 8.2.3. *The stationary points of the $\mathscr{L}$ are solutions to the following equations*

$$\not{D}_B \phi = 0\,; \tag{8.6}$$

$$\tfrac{1}{2}\gamma(*\mathrm{Tr}(F_B)) = \overline{\phi} \otimes \phi - \tfrac{1}{2}|\phi|^2\,. \tag{8.7}$$

Proof. Let $(B, \phi), (b, f) \in \mathscr{C}(M, \sigma)$ and compute

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t=0} \mathscr{L}(B + tb, \phi + tf) = \ & \left.\frac{d}{dt}\right|_{t=0} \left(\frac{1}{8} \int_M \mathrm{Tr}(B + itb - B_0) \wedge \mathrm{Tr}(F_B + itdb + F_{B_0}) \right. \\
& \left. + \frac{1}{2} \int_M \langle \not{D}_B(\phi + tf) + itb \cdot (\phi + tf), \phi + tf \rangle \mathrm{vol} \right) \\
= \ & \frac{i}{4} \int_M b \wedge \mathrm{Tr}(F_B + F_{B_0}) + \frac{i}{4} \int_M \mathrm{Tr}(B - B_0) \wedge db \\
& + \frac{1}{2} \int_M \langle ib \cdot \phi, \phi \rangle \mathrm{vol} + \int_M \mathrm{Re}(\langle \not{D}_B \phi, f \rangle) \mathrm{vol}\,.
\end{aligned}$$

The last term on the right-hand side only vanishes for every f if and only if $\not{D}_B \phi = 0$. Moreover, integrating the second term by parts, we find the remaining equation

$$\frac{i}{2} \int_M (b \wedge \mathrm{Tr}(F_B) + \langle b \cdot \phi, \phi \rangle) = 0\,.$$

Note that if we pick a local orthonormal frame (φ_1, φ_2) for S, we see

$$\langle b \cdot \phi, \phi \rangle = \sum_{i=1,2} \langle \langle \phi, \varphi_i \rangle b \cdot \varphi_i, \phi \rangle$$

$$= \sum_{i=1,2} \langle b \cdot \varphi_i, \langle \varphi_i, \phi \rangle \phi \rangle$$
$$= 2\langle \gamma(b), \overline{\phi} \otimes \phi \rangle \,.$$

Note that the factor $\frac{1}{2}$ in the operator norm is conventional, to preserve norm under the Clifford action. Moreover, because we use the complex linear Hodge $*$-operator and the curvatire is imaginary, we have

$$\int_M b \wedge \mathrm{Tr}(F_B) = -\int_M b \wedge \overline{\mathrm{Tr}(F_B)} = -\int_M \langle b, *\mathrm{Tr}(F_B)\rangle \mathrm{vol}\,,$$

so we conclude

$$\int_M \langle \gamma(b), 2\overline{\phi} \otimes \phi - \gamma(*\mathrm{Tr}(F_B))\rangle \mathrm{vol} = 0\,.$$

Now, b is a one-form, so $\gamma(b)$ is a traceless endomorphism of S, so we see that this vanishes for each $b \in \Omega^1(M;\mathbb{R})$ iff

$$(2\overline{\phi} \otimes \phi - \gamma(*\mathrm{Tr}(F_B)))_0 = 0\,,$$

where $(-)_0$ indicates taking the traceless part. Since $*\mathrm{Tr}(F_B)$ is a one form, $\gamma(*\mathrm{Tr}(F_B))$ is traceless, so we get the second equation

$$\tfrac{1}{2}\gamma(*\mathrm{Tr}(F_B)) = (\overline{\phi} \otimes \phi)_0 = \overline{\phi} \otimes \phi - \tfrac{1}{2}|\phi|^2\,.$$

□

Definition 8.2.4. A stationary point of $\mathscr{L}$ is called an **unperturbed monopole**. The equations in the previous Proposition are the **unperturbed three-dimensional monopole equations**.

Like in the four-dimensional case, we have some curvature estimates for solutions to the monopole equations

Theorem 8.2.5. *Let (M,g) be a closed Riemannian three-manifold with scalar curvature s and let (B,ϕ) be a monopole for (M,g), then*

$$\|\phi\|_\infty^2 \le \max(0, -2\max s)\,.$$

The CSD-functional is not invariant under the action of the full gauge group. The gauge group acts on the connection by $u \cdot B = B - u^{-1}du$. If we define $\alpha_u := \frac{1}{2\pi i}u^{-1}du$, we find

Lemma 8.2.6. *The CSD-functional transforms under the gauge group as*

$$\mathscr{L}(u(B,\phi)) = \mathscr{L}(B,\phi) + 2\pi^2 \int_M \alpha_u \wedge c_1(\det\sigma)\,. \tag{8.8}$$

In particular, the CSD-functional is invariant under the identity component of the gauge group.

The three-dimensional monopole equations are related to the Seiberg-Witten equations on cylinders. If M is a three-manifold with $Spin^c$-structure σ and spinor bundle S, we can equip $M \times \mathbb{R}$ with bundle $S \oplus S$, and define $\gamma : \mathbb{C}l(T(M\times\mathbb{R})) \to \mathrm{End}(S \oplus S)$ by

$$\gamma(\partial_t) := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{8.9}$$

$$\gamma(\pi^* X) := \begin{pmatrix} 0 & -\gamma(X)^* \\ \gamma(X) & 0 \end{pmatrix}, \tag{8.10}$$

where $\pi : M \times \mathbb{R} \to M$ is the projection.

Lemma 8.2.7. *The above defines a* $Spin^c$*-structure* $\pi^*\sigma$ *on* $M \times \mathbb{R}$ *such that the decomposition* $S^+ \oplus S^-$ *agrees with the decomposition* $S \oplus S$ *.*

We get

Theorem 8.2.8. *Let* $(B_t, \phi_t)_{t\in\mathbb{R}} \subseteq \mathscr{C}(M, \sigma)$ *be a smooth one-parameter family of configurations. Then the associated configuration*

$$(B_t + dt, \phi_t) \in \mathscr{C}(M \times \mathbb{R}, \pi^*\sigma)$$

is a solution to the four-dimensional Seiberg-Witten equations if and only if $(B_t, \phi_t)_{t\in\mathbb{R}}$ *solves the downward gradient flow equations for the CSD-functional:*

$$\frac{d}{dt}(B_t, \phi_t) = -\mathrm{grad}(\mathscr{L})(B_t, \phi_t)\,. \tag{8.11}$$

Any configuration on $M \times \mathbb{R}$ can be chosen to have **temporal gauge**, i.e. a gauge such that $(A, \psi) = (B_t + dt, \phi_t)$. On the one hand, a section $\psi \in \Gamma(S^+)$ defines a one-parameter family of sections of S because $S^+ = \pi^*S$, on the other hand, fixing a reference connection A_0 in temporal gauge, we see that $A = A_0 + ib(t) + iadt$, where $b(t)$ is a one-parameter family of one forms on M , and a is a real valued function. If we then pick a gauge u solving $u = a\dot{u}$, we see $u \cdot (A, \psi)$ is in temporal gauge. We see that the remaining gauge freedom is precisely the gauge group of M .

The upshot is that if we find a solution to the Seiberg-Witten equations on a half-cylinder $M \times [0, \infty)$, we can put it in temporal gauge, so the one-parameter family of configurations on M will either diverge in some way as $t \to \infty$, or it will move towards a stationary point of the CSD-functional. The invariant that captures this behaviour is precisely the energy of a four-dimensional configuration:

Theorem 8.2.9. *Let* $(B_t + dt, \phi_t)$ *be a solution to the Seiberg-Witten equations on* $M \times [0, \infty)$ *(in temporal gauge). If the energy*

$$\mathscr{E}(A, \psi) = \int_{M\times[0,\infty)} (|\nabla_A\psi|^2 + |F_A^+|^2 + \tfrac{s}{4}|\psi|^2 + \tfrac{1}{8}|\psi|^4)\mathrm{vol}$$

is finite, then (B_t, ϕ_t) *converges in any Sobolev norm to a monopole* (B, ϕ) *on* M *.*

8.2.2 A few words about asymptotically cylindrical manifolds

This section will be rather sketchy. For the applications we gave at the start of this chapter, we always had manifolds that looked like $M \times \mathbb{R}^+$ away from a compact submanifold (with boundary).

Definition 8.2.10. An **asymptotically cylindrical manifold** is a tuple $(M, g_M, N, g_N, \varphi)$, where (M, g) is an open Riemannian n-manifold, (N, g_N) is a closed Riemannian $(n-1)$-manifold, and φ is an isometry $\varphi : (M \setminus \overline{U}, g_M|_{M\setminus U}) \to (N \times [0, \infty), g_N + dt^2)$, where U is a relatively compact open submanifold of M . We call $(N, g_N) \times [0, \infty)$ the **cylindrical end** of M , we call N the **asymptote** of M and we call U the **bulk** of M .

The "correct" function spaces to study on such manifolds are the spaces $L^{k,p}_\mu$ consisting of functions that decay suitably quickly as $t \to \infty$.

Definition 8.2.11. Let M be an asymptotically cylindrical manifold asymptotic to N. Let $t : M \to \mathbb{R}$ be a smooth function such that $t|_{N\times[0,\infty)} = \pi_{[0,\infty)}$. Let $k \in \mathbb{N}_0$, $p \in [1,\infty)$ and $\mu > 0$, then

$$L^{k,p}_\mu(M) := \{f \in L^{k,p}_{\text{loc}}(M) : \|fe^{t\mu}\|_{k,p} < \infty\}. \tag{8.12}$$

Thus, $L^{k,p}_\mu(M)$ are those functions that converge to 0 faster than $e^{-t\mu}$. Likewise, one can define

Definition 8.2.12. An *asymptotically cylindrical vector bundle* $E \to M$ is a vector bundle E over an asymptotically cylindrical manifold M asymptotic to N together with a vector bundle $F \to N$ and choice of isomorphism $E|_{N\times[0,\infty)} \xrightarrow{\cong} \pi^*F$, where $\pi : N \times [0,\infty) \to N$ is the projection.

Likewise, one can define $L^{k,p}_\mu(E)$ for asymptotically cylindrical vector bundles with asymptotically cylindrical metrics.

One can also define asymptotically cylindrical $Spin^c$-structures, where we note that two isomorphic $Spin^c$-structures need not be isomorphic as asymptotically cylindrical $Spin^c$-structures, since an isomorphism of $Spin^c$ structures over the cylindrical end need not extend to an actual isomorphism of $Spin^c$-structures.

Definition 8.2.13. Let M be an asymptotically cylindrical $Spin^c$ manifold with asymptotically cylindrical $Spin^c$ structure σ asymptotic to (N, σ_N). Then we define

$$\mathscr{C}^{k,p}_\mu(M,\sigma) := \{(A,\psi) \in C(M,\sigma) : \exists (B,\phi) \in \mathscr{C}^{k,p}(N,\sigma|_N) \text{ s.t. } (A,\psi) - \chi(\pi^*(B,\phi)) \in L^{k,p}_\mu\},$$

where $\pi : N \times [0,\infty) \to N$ is the projection and χ is a bump-function supported in $N \times [0,\infty)$ such that $\chi \equiv 1$ on $N \times [1,\infty)$.

I.e., $\mathscr{C}^{k,p}_\mu(M,\sigma)$ is the space of configurations that converge to a configuration on N faster than a fixed exponential.

If we let $\eta \in \Omega^2_{+,c}(M)$ be a compactly supported perturbation, we can consider finite energy $\mathscr{C}_\mu(M,\sigma)$-solutions to the perturbed Seiberg-Witten equations on M with perturbation η. Let $\mathscr{M}_{\eta,\mu}(M,\sigma)$ denote the moduli space of such solutions modulo gauge-transformations u such that u converges faster than $e^{-\mu t}$ to something cylindrical. Since such solutions to the Seiberg-Witten equations always converge to monopoles on the asymptote, we get a canonical map $\partial_\infty : \mathscr{M}_{\eta,\mu}(M,\sigma) \to \mathscr{M}_{\sigma|_N}$, the moduli space of monopoles on N modulo gauge transformations on N. Likewise, let $\mathscr{M}_{\eta,\mu}(M,\sigma,*)$ be the moduli space of irreducible solutions with a fixed value at some point $*$, i.e. this has a residual S^1-action.

The idea is now that if we have two asymptotically cylindrical manifolds M_1, M_2 with the same asymptote N, we can glue $X_1 := M_1 \setminus N \times [r+1,\infty)$ and $X_2 := M_2 \setminus N \times [r+1,\infty)$ together along a time reversing diffeomorphism $N \times (r, r+1) \to N \times (r, r+1)$, such that if we pick r large enough and we pick a basepoint $*$ somewhere in the neck, we can glue $(A_1,\psi_1) \in \mathscr{M}_{\eta_1,\mu}(M_1,\sigma_1,*)$ and $(A_2,\psi_2) \in \mathscr{M}_{\eta_1,\mu}(M_2,\sigma_2,*)$ together whenever $\partial_\infty(A_1,\psi_1) = \partial_\infty(A_2,\psi_2)$, since they almost agree on the neck. Morally, one would be led to a formula like

$$\mathscr{M}_{\eta_1+\eta_2}(X_1 \cup_{N\times(r,r+1)} X_2, \sigma_1 \cup \sigma_2, *) \cong \mathscr{M}_{\eta_1,\mu}(M_1,\sigma_1,*) \times_{\mathscr{M}_{\sigma|_N}} \mathscr{M}_{\eta_1,\mu}(M_2,\sigma_2,*),$$

where $X_1 \cup_{N\times(r,r+1)} X_2$ is a closed Riemannian $Spin^c$ four-manifold. However, such a formula is not true in general.

Bibliography

[BH21] A. Behzadan and M. Holst. “Multiplication in Sobolev spaces, revisited”. In: *Arkiv för Matematik* 59.2 (Oct. 2021).

[Mor96] John W. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*. Vol. 44. Mathematical Notes. Princeton University Press, 1996. ISBN: 0-691-02597-5.

[Nic00] Liviu I. Nicolaescu. *Notes on Seiberg-Witten Theory*. Vol. 28. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2000.

Appendix A

Proper group actions

Definition A.0.1. Let X be a topological space and let G be a topological group. Assume that $G \curvearrowright X$ continuously. We say that the action is *proper* if the shear map $S\colon G \times X \to X \times X, (g, x) \mapsto (x, g \cdot x)$ is proper.

We have the following alternative characterizations.

Proposition A.0.2. *Let $G \curvearrowright X$ be a continous group action. Assume that G and X are metrizable. Then the following are equivalent:*

1. *The action is proper.*
2. *For all sequences $(x_n) \subset X$ and $(g_n) \subset G$ such that $x_n \to x$ and $g_n \cdot x_n \to y$, there is a convergent subsequence of (g_n).*
3. *For all compact $K \subset X$, the set $G_K = \{g \in G \mid g \cdot K \cap K \neq \emptyset\}$ is compact.*

Proof. Note that for metrizable space the notions of compactness and sequential compactness coincide. Moreover, since subspaces are metrizable as well, this property is hereditary. We will use this fact in the proof.

(1 $\Rightarrow$ 2) Suppose that the action is proper. Let $(x_n) \subset X$ and $(g_n) \subset G$ be sequences such that $x_n \to X$ and $g_n \cdot x_n \to y$. Then for all n we have $S(g_n, x_n) = (x_n, g_n \cdot x_n)$, so by assumption $S(g_n, x_n) \to (x, y)$. So the set $L = \{S(g_n, x_n) \mid n \in \mathbb{N}\} \cup \{(x, y)\}$ is compact. Since S is proper by assumption, the set $S^{-1}(L) \subset G \times X$ is compact. So $\mathrm{pr}_G(S^{-1}(L)) \subset G$ is compact and $(g_n) \subset \mathrm{pr}_G(S^{-1}(L))$. So by (sequential) compactness, (g_n) has a convergent subsequence.

(2 $\Rightarrow$ 3) Let $K \subset X$ be compact. Let $(g_n) \subset G_K$ be a sequence. Then for each $n \in \mathbb{N}$ we can choose an $x_n \in K$ such that $g_n \cdot x_n \in K$. Then we can extract a subsequence of x_n and g_n which we denote by the same symbol such that (x_n) and $(g_n \cdot x_n)$ converge, using the compactness of K. Then by 2 we have a convergent subsequence of g_n. So G_K is sequentially compact, and so also compact.

(3 $\Rightarrow$ 1) We have to show that the shear map $S\colon G \times X \to X \times X$ is proper. Let $K \subset X \times X$ be compact. Then $K_1 = \mathrm{pr}_1(K)$ and $K_2 = \mathrm{pr}_2(K)$ are compact, hence $L = K_1 \cup K_2$ is compact. Then $L \times L$ is also compact and $K \subset L \times L$. We claim that $S^{-1}(L \times L) \subset G_L \times L$.

Let $(g, x) \in S^{-1}(L \times L)$. Then $(x, g \cdot x) \in L \times L$, so $x \in L$ and $g \cdot x \in L \cap g \cdot L$. So $g \in G_L$. By assumption G_L is compact, so $G_L \times L$ is compact. Finally, by continuity, $S^{-1}(K) \subset S^{-1}(L \times L) \subset G_L \times L$ is a closed subset. So $S^{-1}(K)$ is also compact. So S is proper. $\square$

We have used the following lemma in the proof.

Lemma A.0.3. *Let X be a topological space. Let $(x_n) \subset X$ be a sequence. Assume that $x_n \to x$. Then $L = \{x_n \mid n \in \mathbb{N}\} \cup \{x\}$ is compact.*

Proof. Let $\mathfrak{U}$ be an open cover of L. Then there is a $U_0 \in \mathfrak{U}$ such that $x \in U_0$. Since $x_n \to x$, there exists an $N \in \mathbb{N}$ such that for $n \geq N$ we have $x_n \in U_0$. Now, for $0 \leq k \leq N-1$, we pick $U_{k+1} \in \mathfrak{U}$ such that $x_k \in U_{k+1}$. Then $\{U_0, U_1, \ldots, U_N\}$ is a finite subcover. □

Lemma A.0.4. *Let $G \curvearrowright X$ be a continuous group action. Then the quotient map $q\colon X \to X/G$ is open.*

Proof. Let $U \subset X$ be open. Then

$$q^{-1}(q(U)) = \bigcup_{g \in G} g \cdot U$$

which is open. So by the quotient map property, $q(U)$ is open. □

Lemma A.0.5. *Let $G \curvearrowright X$ be a proper group action. Let $R \subset X \times X$ be the equivalence relation on X induced by the action. Then R is closed as a subset.*

Proof. Let $S\colon G \times X \to X \times X$ be the shear map of the action. We claim that $R = S(G \times X)$, which is then closed since the image of a proper map is closed. We have $(x, y) \in R$ iff $x \sim_G y$ iff there exists a $g \in G$ such that $y = g \cdot x$ iff there exists a $g \in G$ such that $S(g, x) = (x, y)$ iff $(x, y) \in S(G \times X)$. So indeed $R = S(G \times X)$. □

Proposition A.0.6. *Let $G \curvearrowright X$ be a continuous proper group action. Assume that X and G are Hausdorff. Then the quotient space X/G is Hausdorff.*

Proof. Denote the quotient map by $q\colon X \to X/G$. Let $q(x), q(y) \in X/G$. Assume that $q(x) \neq q(y)$. Then $(x, y) \notin R$. Since R is closed, there are open neighbourhoods $U, V \subset X$ of x and y respectively, such that $U \times V \cap R = \emptyset$. Then $q(U)$ and $q(V)$ are open neighbourhoods of $q(x)$ and $q(y)$ respectively. Suppose that $q(z) \in q(U) \cap q(V)$. Then there are $g, h \in G$ such that $g \cdot z \in U$ and $h \cdot z \in V$, so $(g \cdot z, h \cdot z) \in U \times V$ and $(g \cdot z, h \cdot z) \in R$, since $g \cdot z = gh^{-1} \cdot (h \cdot z)$. This is a contradiction, so $q(U) \cap q(V) = \emptyset$. So X/G is Hausdorff. □